494/598 Fall 2008

ASSIGNMENT 4

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are said to be *equivalent* if there are positive constants m and M such that

$$m\,\|x\|_1 \leq \|x\|_2 \leq M\,\|x\|_1$$

for all $x \in X$.

In class we will show that the relation $\sim$ defined by $\|\cdot\|_1 \sim \|\cdot\|_2$ is an equivalence relation on the set of norms on X. For that we will need to show reflexivity, symmetry, and transitivity.

1. Show that equivalent norms on X define the same 'topology'. In other words, prove that a set which is open under some norm is open under any equivalent norm.

Recall that we discussed Fatou's Lemma.

2. Construct a counter example that if *liminf* is replaced with *limsup*, then the resulting statement of Fatou's Lemma does not hold.

3. Show that the following "Reverse Fatou's Lemma" holds: If there exists an integrable function g (this means that $\int |g| < +\infty$) such that $g_n \leq g$ for all n, then

$$\int \limsup_{n\to\infty} g_n \geq \limsup_{n\to\infty} \int g_n.$$

Hint: Apply Fatou's lemma to the non-negative sequence given by $g - g_n$.