ASSIGNMENT 3: KEY

1. Let $N \in \mathbb{N}$ be even with $N = 2M$, and suppose $z \in \ell^2(\mathbb{Z}_N)$. Define $u, v \in \ell^2(\mathbb{Z}_M)$ by
$$u(k) = z(2k), \quad k = 0, 1, ..., M-1$$
and
$$v(k) = z(2k+1), \quad k = 0, 1, ..., M-1.$$
Show that for $n = 0, ..., M-1$,
$$\hat{z}(n) = \hat{u}(n) + e^{-2\pi i n/N}\hat{v}(n), \tag{1}$$
and for $n = M, ..., 2N-1$, if we set $l = n - N$, then
$$\hat{z}(n) = \hat{u}(l) - e^{-2\pi i l/N}\hat{v}(l). \tag{2}$$
<u>Solution</u>: Let $n \in \mathbb{Z}_N$ and recall that $M = N/2$. Then
$$\begin{aligned}
\hat{z}(n) &= \sum_{m=0}^{N-1} z(k)\, e^{-2\pi i m\, n/N} \\
&= \sum_{k=0}^{M-1} z(2k)\, e^{-2\pi i(2k)n/(2M)} + \sum_{k=0}^{M-1} z(2k+1)\, e^{-2\pi i(2k+1)n/(2M)} \\
&= \sum_{k=0}^{M-1} u(k)\, e^{-2\pi i k n/M} + e^{-2\pi i n/2M} \sum_{k=0}^{M-1} v(k)\, e^{-2\pi i k\, n/M} \\
&= \hat{u}(n) + e^{-2\pi i n/N}\hat{v}(n),
\end{aligned}$$
giving the first equality. If $M \le n \le 2N-1$, then, setting $l = n - N$, $e^{-2\pi i n/2N} = e^{-\pi i - 2\pi i l/2N}$, so the second equality follows from the fact that $e^{-\pi i} = -1$.

2. In the previous exercise we showed that we can compute the DFT of a vector of length N with N^2 complex multiplications (this means that $\hat{z} = W_N z$ and since W_N is $N \times N$ matrix, one would need to perform N^2 (complex) multiplications).

   Show that if N is a power of 2 (denote $N = 2^n$), then we can iterate this procedure to compute the DFT of a vector of length N in at most $\frac{1}{2}N\log_2 N$ complex multiplications.

   <u>Solution</u>:
   - *Base Step:* Let $N = 2$. Then, since the two vector in the Fourier basis are by definition $(1,1)$ and $(1,-1)$, it follows from the definition of the inner product that $\widehat{(a,b)} = (a+b, a-b)$, which requires no complex multiplications to compute.
   - *Inductive Step:* Suppose that the result holds for $N = 2^{k-1} \ge 2$ Using the algorithm in (1), we can compute the DFT for a vector of length N in $\frac{1}{2}N + 2M$ computations, where $M \le \frac{1}{4}Nlog_2(\frac{1}{2}N)$ is the number of computations required co compute the DFT of a vector of length $\frac{1}{2}N$. Since, by the definition of the logarithm, $\log_2(N/2) = log_2 N - 1$, It follows that we can compute the DFT of a vector of length N in at most
$$\frac{1}{2}N + \frac{1}{4}Nlog_2(\frac{1}{2}N) = \frac{1}{2}[N + \frac{1}{2}(log_2(N) - 1)] = \frac{1}{2}log_2(N).$$

3. In Lecture Notes (Week 3), we claimed that the properties of the higher-dimensional DFT followed from the fact that it is simply a component-wise iteration of the one-dimensional DFT. We also claimed that these properties could be derived directly from the fact that if $F_\beta(\alpha) = e^{2\pi i \alpha\cdot\beta/N}$, where α, β, N are multi-indexes in $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times ... \times \mathbb{Z}_{N_d}$, then the F_β's form an orthogonal basis for $\ell^2(\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times ... \times \mathbb{Z}_{N_d})$. Prove this by showing that

$$\langle F_\beta, F_\gamma \rangle = \begin{cases} 0 & \beta \neq \gamma, \\ N_1 \cdot N_2 \cdot ... \cdot N_M & \beta = \gamma. \end{cases}$$

<u>Solution</u>: We will write out the 3d case:

$$\begin{aligned} < F_\beta, F_\gamma > &= \sum_{\alpha_1=0}^{N_1-1} \sum_{\alpha_2=0}^{N_2-1} \sum_{\alpha_3=0}^{N_3-1} F_{\beta_1,\beta_2,\beta_3}(\alpha_1,\alpha_2,\alpha_3)\, \overline{F}_{\gamma_1,\gamma_2,\gamma_3}(\alpha_1,\alpha_2,\alpha_3) \\ &= \sum_{\alpha_1=0}^{N_1-1} \sum_{\alpha_2=0}^{N_2-1} \sum_{\alpha_3=0}^{N_3-1} e^{2\pi i(\alpha_1\beta_1/N_1+\alpha_2\beta_2/N_2+\alpha_3\beta_3/N_3)}\, e^{-2\pi i(\alpha_1\gamma_1/N_1+\alpha_2\gamma_2/N_2+\alpha_3\gamma_3/N_3)} \\ &= \sum_{\alpha_1=0}^{N_1-1} e^{2\pi i\alpha_1(\beta_1-\gamma_1)/N_1} \sum_{\alpha_2=0}^{N_2-1} e^{2\pi i\alpha_2(\beta_2-\gamma_2)/N_2} \sum_{\alpha_3=0}^{N_3-1} e^{2\pi i\alpha_3(\beta_3-\gamma_3)/N_3}. \end{aligned}$$

If the multi-index $\beta \neq \gamma$, i.e., $\beta_i \neq \gamma_i$ for all i, then a typical sum in the last expression

$$\sum_{\alpha=0}^{N-1} e^{2\pi i\alpha(\beta-\gamma)/N} = \sum_{\alpha=0}^{N-1} \left(e^{2\pi i(\beta-\gamma)/N}\right)^\alpha$$

is a finite geometric series

$$\sum_{k=0}^{N-1} r^k = \frac{1-r^N}{1-r} \quad \text{with} \quad r = e^{2\pi i(\beta-\gamma)/N}$$

but since $r^N = e^{2\pi i(\beta-\gamma)/N}N = 1$, such a sum would be equal to zero. Thus, $< F_\beta, F_\gamma >= 0$ if $\beta \neq \gamma$.

Now if the multi-index $\beta \neq \gamma$, then $e^0 = 1$ and

$$< F_\beta, F_\gamma >= \sum_{\alpha_1=0}^{N_1-1} 1 \sum_{\alpha_1=0}^{N_2-1} 1 \sum_{\alpha_3=0}^{N_3-1} 1 = N_1 \cdot N_2 \cdot N_3.$$

4. Write the Fourier basis for $l^2(\mathbb{Z}_4)$. What is the Euclidean to Fourier change-of-basis matrix W_4 ?

<u>Solution</u>:

$$\mathcal{F} = \{F_1, F_2, F_3, F_4\},$$

where $F_1 = (1,1,1,1)^t$, $F_2 = (1,i,-1,-i)^t$, $F_3 = (1,-1,1,-1)^t$, $F_4 = (1,-i,-1,i)^t$.

$$W_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

5. (i) Find a 2×2 unitary matrix U that is not the identity.
Hint: recall that a matrix is unitary if and only if it's rows form an orthonormal basis for $\mathbb{R}^n$ if and only if its columns do. So you need only find an orthonormal basis for $\mathbb{R}^2$ which is not the standard basis. Why not standard basis? Because it will give you the identity matrix!

<u>Solution</u>: The vectors $u = \frac{1}{\sqrt{2}}(1,1)$ and $\frac{1}{\sqrt{2}}(1,-1)$ are orthonormal in $\mathbb{R}^2$ (they form the Fourier basis for $l^2(\mathbb{Z}_2)$.) It follows that the matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

is unitary. Of course, in this case the matrix has real entries, but this need not be the case.
Another example is rotation in $\mathbb{R}^2$. If we rotate counterclockwise a vector to angle φ, then the matrix of rotation R_φ is

$$R_\phi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

Check that it is unitary.
And of course, there are many more other unitary matrices!

(ii) Find a 2×2 normal matrix which is not unitary.
Hint: recall that a matrix A is normal if and only if it is unitarily diagonalizable, i.e., if there is a diagonal matrix D and a unitary matrix U such that $A = U^*DU$.
<u>Solution</u>: Let $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. Then by direct computation, with U as in the previous exercise, we have

$$U^*DU = \frac{1}{2} \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}.$$

Of course, in this case U^*DU is symmetric, since all the entries are real, and it is clear from the definition that any symmetric matrix will be normal (this could give an easier answer to this question than the hint led to, since any symmetric 2×2 matrix would work).