ASSIGNMENT 3

(CORRECTED!)

1. Let $N \in \mathbb{N}$ be even with $N = 2M$, and suppose $z \in \ell^2(\mathbb{Z}_N)$. Define $u,\, v \in \ell^2(\mathbb{Z}_M)$ by

$$u(k) = z(2k), \quad k = 0, 1, ..., M-1$$

and

$$v(k) = z(2k+1), \quad k = 0, 1, ..., M-1.$$

Show that for $n = 0, ..., M-1$,

$$\hat{z}(n) = \hat{u}(n) + e^{-2\pi i n/N}\hat{v}(n), \tag{1}$$

and for $n = M, ..., 2N-1$, if we set $l = n - N$, then

$$\hat{z}(n) = \hat{u}(l) - e^{-2\pi i l/N}\hat{v}(l). \tag{2}$$

Hint: First, prove (1) *by writing the definition of* $\hat{z}$ *and splitting the initial sum into 2 sums (odd and even indexes) and enumerating each of the sums using the fact that* $M = N/2$*, then substituting the definitions of* u *and* v *into the sums and simplifying. In a similar fashion obtain the second inequality for* $M \leq n \leq N-1$ *by reindexing* $n = l - M$*, so* $0 \leq l \leq M-1$*.*

2. In the previous exercise we showed that we can compute the DFT of a vector of length N with N^2 complex multiplications (this means that $\hat{z} = W_N z$ and since W_N is $N \times N$ matrix, one would need to perform N^2 (complex) multiplications).

Show that if N is a power of 2 (denote $N = 2^n$), then we can iterate this procedure to compute the DFT of a vector of length N in at most $\frac{1}{2}N\log_2 N$ complex multiplications.

Hint: Use induction on the power of 2, i.e., on n *where* $N = 2^n$*. Use a vector of length 2 for the base step, that is,* $n = 1$*. Show, either directly using the definition of the Fourier transform on* $\ell^2(Z_2)$*, or indirectly using Exercise 1 and the Fourier transform on* $\ell^2(Z_1)$*, that if* $z = (z(0), z(1))$*, then* $\hat{z} = (z(1) + z(2), z(1) - z(2))^t$*. This does not require any complex multiplications, so* $\#_2 = 0$ *which is consistent with the formula* $\leq \frac{1}{2}N\log_2 N$ *for* $N = 2$*. For the induction step* $n-1$*, or for* $N/2 = 2^{n-1}$*, assume that it is known that*

$$\#_{N/2} \leq \frac{1}{2}\left(\frac{N}{2}\right)\log_2\left(\frac{N}{2}\right). \tag{3}$$

Explain why Exercise 1 shows that

$$\#_N = 2\,\#_{N/2} + N/2. \tag{4}$$

Use equations (3) *and* (4) *to show that*

$$\#_N \leq \frac{1}{2}N\log_2 N.$$

3. In Lecture Notes (Week 3), we claimed that the properties of the higher-dimensional DFT followed from the fact that it is simply a component-wise iteration of the one-dimensional DFT. We also claimed that these properties could be derived directly from the fact that if

$F_\beta(\alpha) = e^{2\pi i \alpha\cdot\beta/N}$, where α, β, N are multi-indexes in $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times ... \times \mathbb{Z}_{N_d}$, then the F_β's form an orthogonal basis for $\ell^2(\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times ... \times \mathbb{Z}_{N_d})$. Prove this by showing that

$$\langle F_\beta, F_\gamma\rangle = \begin{cases} 0 & \beta \neq \gamma, \\ N_1 \cdot N_2 \cdot ... \cdot N_M & \beta = \gamma. \end{cases}$$

Hint: You can do this exercise only in 2 or 3 dimensions. First, write out what multi-indexes α, β and N mean: for example, in 3d $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Also write out explicitly what F_β is. For example, in 3d $F_\beta(\alpha) = e^{2\pi i(\alpha_1\beta_1/N_1+\alpha_2\beta_2/N_2+\alpha_3\beta_3/N_3)}$. Lastly, write out what the inner product is in several dimensions. For example, in 3d

$$< F_\beta, F_\gamma >= \sum_{\alpha_1=0}^{N_1-1}\sum_{\alpha_2=0}^{N_2-1}\sum_{\alpha_3=0}^{N_3-1} F_{\beta_1,\beta_2,\beta_3}(\alpha_1,\alpha_2,\alpha_3)\,\bar{F}_{\gamma_1,\gamma_2,\gamma_3}(\alpha_1,\alpha_2,\alpha_3).$$

Now substitute the expression for F's and calculate the sum, if needed see Lecture Notes week 3 formula (2.1), or recall finite geometric series.

4. Write the Fourier basis for $l^2(\mathbb{Z}_4)$. What is the Euclidean to Fourier change-of-basis matrix W_4 ?

5.

(a) Find a 2×2 unitary matrix U that is not the identity.
Hint: recall that a matrix is unitary if and only if it's rows form an orthonormal basis for $\mathbb{R}^n$ if and only if its columns do. So you need only find an orthonormal basis for $\mathbb{R}^2$ which is not the standard basis. Why not standard basis? Because it will give you the identity matrix!

(b) Find a 2×2 normal matrix which is not unitary.
Hint: recall that a matrix A is normal if and only if it is unitarily diagonalizable, i.e., if there is a diagonal matrix D and a unitary matrix U such that $A = U^*DU$.