ASSIGNMENT 2: KEY

Let

$$A = \begin{pmatrix} 4 & 0 & 5 \\ 3 & 1 & 6 \\ 2 & 0 & 7 \end{pmatrix}.$$

(a) Find A^{-1}.

(*You can use any method you know to find this inverse, for example, Gaussian elimination, or any other you may have learned in a Linear Algebra class.*)

<u>Solution</u>: Performing Gauss-Jordan elimination on the matrix

$$\left(\begin{array}{ccc|ccc} 4 & 0 & 5 & 1 & 0 & 0 \\ 3 & 1 & 6 & 0 & 1 & 0 \\ 2 & 0 & 7 & 0 & 0 & 1 \end{array}\right)$$

to make the left half into the 3×3 identity, and taking A^{-1} to be what results in the right half (this process, or something similar, should be familiar from a course in elementary Linear Algebra), we find that

$$A^{-1} = \frac{1}{18}\begin{pmatrix} 7 & 0 & -5 \\ -9 & 18 & -9 \\ -2 & 0 & 4 \end{pmatrix}.$$

(b) Find the eigenvalues and eigenvectors of A.

<u>Solution</u>: Expanding the determinant of $A - \lambda I$ along the first minor, the characteristic polynomial of A is

$$\begin{aligned} (4-\lambda)(1-\lambda)(7-\lambda) + 5(-2(1-\lambda)) &= (1-\lambda)(18 - 11\lambda + \lambda^2) \\ &= (1-\lambda)(9-\lambda)(2-\lambda), \end{aligned}$$

so that the eigenvalues of A are 1, 2, and 9. Direct solution of the systems of equations defined by $(A-\lambda I)x = 0_{\mathbb{R}^3}$ shows that the eigenspaces of 1, 2, and 9, are the spaces spanned by the vectors $(0, 1, 0)$, $(5, -3, -2,)$, and $(8, 9, 8)$, respectively.

(c) Diagonalize A. That is, find matrices P and D such that $A = PDP^{-1}$.

(*You need not compute P^{-1}, it is straightforward, but the fractions get a bit ugly*).

<u>Solution</u>: By the definition of an eigenvector-eigenvalue pair, and computing each column independently, we have that $AP = PD$, and hence, $A = PDP^{-1}$, if D is the diagonal matrix containing the eigenvalues of A, and P is the matrix whose columns are the corresponding eigenvectors, in the same order. Hence, we take

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

and

$$P = \begin{pmatrix} 5 & 0 & 8 \\ -3 & 1 & 9 \\ -2 & 0 & 8 \end{pmatrix}.$$

(d) Use the diagonalization from part (b) to find A^{10}.

(*Your solutions may be left in terms of P and P^{-1}, since we did not compute explicitly P^{-1}, but not in terms of D. Hint: use $A = PDP^{-1}$ and then cancelations between P and P^{-1}.*)

<u>Solution</u>: We have

$$A^{10} = AAA...A = PDP^{-1}PDP^{-1}PDP^{-1}...PDP^{-1} = PDDD...DP^{-1},$$

so the solution is $PD^{10}P^{-1}$, where

$$D^{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & 9^{10} \end{pmatrix}.$$

(e) Find a (possibly complex-valued) matrix C such that $C^2 = A$.
(*Use hints from the previous part.*)

<u>Solution</u>: Similarly to the previous solution, if $C = PD^{1/2}P^{-1}$, where

$$D^{1/2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

then we will have

$$C^2 = PD^{1/2}P^{-1}PD^{1/2}P^{-1} = PD^{1/2}D^{1/2}P^{-1} = PDP^{-1} = A.$$

(f) Calculate the conjugate transpose of

$$\begin{pmatrix} 1 & i & 1 \\ 0 & 1 & 0 \\ i & 1 & -i \end{pmatrix}.$$

<u>Solution</u>: The conjugate transpose is

$$\begin{pmatrix} 1 & 0 & -i \\ -i & 1 & 1 \\ 1 & 0 & i \end{pmatrix}.$$