ASSIGNMENT 13: INVERSION OF THE RADON TRANSFORM (CONT.)

Let's recall that in $2d$ the Radon transform $\mathcal{R}f$ can be inverted to obtain the function f by means of the filtered back projection

$$f = \mathcal{R}^* \circ (\mathcal{G}\mathcal{R})f, \tag{1}$$

where $\mathcal{G}\mathcal{R}$ is the '*Filter*' operator and the adjoint $\mathcal{R}^*$ is the back projection operator. We have also shown that the Filter operator $\mathcal{G}\mathcal{R}$ is the Hilbert transform of the derivative of $\mathcal{R}f$, thus, the equation (1) can be written as

$$f = \mathcal{R}^* \circ \mathcal{H}(\partial_t \mathcal{R})f. \tag{2}$$

Alternatively, we have shown that using the fractional powers of the laplacian, i.e., $(-\Delta)^s$, we can compute the inverse Radon transform of f as follows

$$f = \frac{1}{4\pi}(-\Delta)^{1/2}\left(\mathcal{R}^*\mathcal{R}f\right). \tag{3}$$

In the last assignment we considered the $3d$ version of the Radon transform and, in particular, have shown that the Filter operator can be represented by a second partial derivative (in the affine coordinate) of the Radon transform ($\mathcal{G}\mathcal{R} = -\partial_t^2 \mathcal{R}f$), and hence, the reconstruction in the $3d$ case, similar to the $2d$ inversion formula (2) is

$$f = \mathcal{R}^*\left(-\partial_t^2 \mathcal{R}f\right), \tag{4}$$

or,

$$f(\vec{x}) = \mathcal{R}^*\left(-\partial_t^2 \mathcal{R}f\right)(\vec{x}), \quad \vec{x} \in \mathbb{R}^3.$$

In this assignment we will obtain an alternative inversion formula in $3d$ which is analogous to (3). In fact, such an alternative formula can be obtained for all dimensions, see a remark afterwards.

Problem 1. Prove

$$f = \frac{1}{8\pi^2}(-\Delta)(\mathcal{R}^*\mathcal{R}f).$$

This equation is written in a way to resemble the one in (3).

(*You may solve it however you want, one possibility is to follow the steps on the opposite page.*)

Remark. The alternative inversion formula for the Radon Transform on $\mathbb{R}^d$ is

$$f = \frac{1}{2(2\pi)^{d-1}}(-\Delta)^{(d-1)/2}(\mathcal{R}^*\mathcal{R}f),$$

where $(-\Delta)^s$ is defined as ($\vec{x}, \vec{\xi} \in \mathbb{R}^d$)

$$\widehat{(-\Delta)^s f}(\vec{\xi}) = |\vec{\xi}|^{2s}\, \hat{f}(\vec{\xi}), \quad \text{or} \quad (-\Delta)^s f(\vec{x}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\vec{\xi}|^{2s}\, \hat{f}(\vec{\xi})\, e^{i\langle \vec{x},\vec{\xi}\rangle}\, d\vec{\xi}.$$

Solution Sketch

a. Show, using the Central Slice Theorem and the definition of $\mathcal{R}^*$, that
$$\mathcal{R}^*\mathcal{R}f(\vec{x}\,) = \int_{S^2}\int_{-\infty}^{\infty} \hat{f}(r\vec{\omega}\,)e^{2\pi i r\,\vec{x}\cdot\vec{\omega}}\,dr d\sigma(\vec{\omega}).$$

b. Plug the identity above into the formula $\frac{1}{8\pi^2}(-\Delta)(\mathcal{R}^*\mathcal{R}f)$.

c. Take the Laplace operator $(-\Delta)$ inside the integration (Justify it using one of the results from the notes on properties of the Lebesgue integral). Then, evaluate the operator, which is really a sum of partial derivatives, directly.

d. Rewrite the integral, using the property of the Radon Transform that it is even, then multiplying by an appropriate constant and taking the integral in the affine parameter from 0 to ∞.

e. Now evaluate the integral to get the result.