ASSIGNMENT 12: KEY

Problem 1. Describe and draw the following planes:
(a) $l_{0,(1,0,0)}$ or $\langle \vec{x}, (1,0,0)\rangle = 0$,
(b) $l_{0,(0,-1,0)}$ or $\langle \vec{x}, (0,-1,0)\rangle = 0$,
(c) $l_{-2,(1,0,0)}$ or $\langle \vec{x}, (1,0,0)\rangle = -2$.

<u>Solution:</u>
(a) The $y - z$ plane.
(b) The $x - z$ plane.
(c) The $y = z$ plane, translated two units in the negative x direction.

Problem 2. Write the plane passing through the following 3 points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ in the form $\langle \vec{x}, \vec{\omega}\rangle = t$ (i.e., find t and $\vec{\omega}$, and then find θ and φ such that $\vec{\omega} = \vec{\omega}(\theta, \varphi)$).

<u>Solution:</u> The easiest way to proceed is to find the intersection point of the plane and the ray defined by $\vec{\omega}$ in $\mathbb{R}^3$ first, and then derive the solution. We know from geometric considerations that the plane is normal to the (non-unit) vector (1,1,1), so the intersection point (x, y, z) will satisfy the equations

$$x = y = z$$

and

$$x + y + z = 1.$$

The solution, then is $(1/3, 1/3, 1/3)$. Since we need $\vec{\omega}$ to be a unit vector, the plane given is $l_{r,\vec{\omega}}$, where $\omega = \sqrt{3}/3(1,1,1)$ and $r^2 = 3(1/3)^2 = 1/3$, so $r = \frac{1}{\sqrt{3}}$.

Expressed in terms of θ and φ, $\vec{\omega}$ is $\vec{\omega}(\cos^{-1}(1/\sqrt{3}), \pi/4)$, where the second argument is obvious from geometric considerations and the fist comes from solving for θ in the formula $z = r\cos\theta$, where z is 1/3 and $r = 1/\sqrt{3}$.

Problem 3. Prove the 3d Central Slice Theorem for $f \in L^1(\mathbb{R}^3)$:

$$\widetilde{\mathcal{R}f}(r, \vec{\omega}) \equiv \int_{-\infty}^{\infty} \mathcal{R}f(t, \vec{\omega})\, e^{-irt}\, dt = \hat{f}(r\vec{\omega}).$$

<u>Solution:</u> Fix orthogonal unit vectors $\vec{u}_1$ and $\vec{u}_2$ perpendicular to $\vec{\omega}$. Below, we use an orthogonal change of variables $x = t\vec{\omega} + s_1\vec{u}_1 + s_2\vec{u}_2$ in the second line, along with the fact that, given that change of variables, we have $r\vec{\omega} \cdot \vec{x} = rt$.

$$\begin{aligned}\int_{\mathbb{R}^3} f(\vec{x}) e^{-ir\vec{\omega}\cdot\vec{x}} dx &= \int_{\mathbb{R}} \int_{\mathbb{R}^2} f(t\vec{\omega} + s_1\vec{u}_1 + s_2\vec{u}_2) e^{-irt} ds_1 ds_2 dt \\ &= \int_{\mathbb{R}} \mathcal{R}f(t, \omega) e^{-irt} dt,\end{aligned}$$

The same proof, incidentally, works for the unitary Fourier transform.

Problem 4. Show that the Radon adjoint on $\mathbb{R}^3$ is, in fact, the adjoint of the Radon Transform. For simplicity, assume that f is bounded and rapidly decreasing on $\mathbb{R}^3$ and h is bounded and rapidly decreasing in the affine parameter on $\mathbb{R} \times S^2$. Show that

$$\langle \mathcal{R}f, h\rangle_{L^2(\mathbb{R}\times S^2)} = \langle f, \mathcal{R}^* h\rangle_{L^2(\mathbb{R}^3)}\,.$$

Solution: Just for demonstration, we provide a proof for the general d-dimensional proof here. The special case $d = 3$ is proved the same way as the general case.

Between the second and third lines below, we make a change of variables $\vec{x} = t\vec{\omega} + s_1\vec{u}_1 + \ldots s_{d-1}\vec{u}_{d-1}$, so that $\vec{x}\cdot\vec{\omega} = t$. In the fourth line line we interchange order of integration between $\sigma(\vec{\omega})$ and the other variables, which we can do using Fubini's theorem and the hypotheses that f and g are L^1:

$$\begin{aligned}
\langle \mathcal{R}f, g\rangle_{\mathbb{R}\times S^{d-1}} &= \int\limits_{S^{d-1}} \int_{-\infty}^{\infty} \mathcal{R}f(t,\omega)\bar{g}(t,\omega)dtd\sigma(\omega)\\
&= \int\limits_{S^{d-1}} \int_{-\infty}^{\infty} \int\limits_{\mathbb{R}^{d-1}} f(t\omega + s_1u_1 + \ldots s_{d-1}u_{d-1})\bar{g}(t,\omega)dtds_1\ldots ds_{d-1}d\sigma(\omega)\\
&= \int\limits_{S^{d-1}} \int\limits_{\mathbb{R}^d} f(x)\bar{g}(x\cdot\omega,\omega)dxd\sigma(\omega)\\
&= \int\limits_{\mathbb{R}^d} f(x) \int\limits_{S^{d-1}} \bar{g}(x\cdot\omega,\omega)d\sigma(\omega)dx\\
&= \int\limits_{\mathbb{R}^d} f(x)\overline{\mathcal{R}^* g}(x)dx\\
&= \langle f, \mathcal{R}^* g\rangle_{\mathbb{R}^d}
\end{aligned}$$

Problem 5. Show that the *Filter* part of the Radon inversion formula is an affine second derivative of the Radon transform:

$$(\mathcal{G}\mathcal{R})f(t,\vec{\omega})|_{t=\langle\vec{x},\vec{\omega}\rangle} \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{\mathcal{R}f}(r,\vec{\omega})\, r^2\, e^{ir\langle\vec{x},\vec{\omega}\rangle}\, dr = -\frac{\partial^2}{\partial t^2}\mathcal{R}f(t,\vec{\omega})|_{t=\langle\vec{x},\vec{\omega}\rangle}.$$

Solution:

Fix ω, and let $g(t) = \mathcal{R}f(t,\vec{\omega})$ to simplify notation. Then we have $\widetilde{\mathcal{R}f}(r,\vec{\omega}) = \hat{g}(r)$, and our goal is to show that

$$\mathcal{F}^{-1}\big(r^2\hat{g}(r)\big)(t) = -\frac{\partial^2}{\partial t^2}g(t),$$

which is equivalent to

$$\mathcal{F}\left(\frac{\partial^2}{\partial t^2}g(t)\right) = (ir)^2\hat{g}(r).$$

This is simply the formula relating the Fourier transform to derivatives, gotten by applying the first order derivative twice, under the assumption that the function is twice weakly differentiable with both the first and second order derivatives L^1.