ASSIGNMENT 12: THE RADON TRANSFORM IN 3D

In this assignment we will define the Radon transform in 3 dimensions and examine its properties. First, we review the polar coordinates. We write $\vec{x} \in \mathbb{R}^3$ in spherical coordinates, $\vec{x} = (x_1, x_2, x_3)$ with

$$\begin{cases} x_1 &= r\cos\varphi\sin\theta, \\ x_2 &= r\sin\varphi\sin\theta, \\ x_3 &= r\cos\theta. \end{cases}$$

Here, φ is the azimuthal angle (in xy plane, $0 \leq \varphi < 2\pi$) and θ is the angle from the z axis ($0 \leq \theta \leq \pi$). In particular, if ω is a unit vector in $\mathbb{R}^3$, then then we can think of $\vec{\omega}$ as depending on two parameters and write $\vec{\omega} = \vec{\omega}(\theta, \varphi) \in S^2$, where the 2 indicates that there are two degrees of freedom in the set of unit vectors in $\mathbb{R}^3$. Since every vector $\vec{x} \in \mathbb{R}^3$ (except 0) has a unique representation $\vec{x} = r\vec{\omega}$ where $r \geq 0$ and $\vec{\omega} \in S^2$ with $\|\omega\| = 1$, we have $\mathbb{R}^3 = \mathbb{R}^+ \times S^2$.

Recall that $d\vec{x} = dx_1\,dx_2\,dx_3$ in rectangular coordinates and $d\vec{x} = r^2 \sin\theta\,dr\,d\theta\,d\varphi$ in spherical coordinates. The last expression can be also viewed as $d\vec{x} = r^2 dr\,d\sigma(\vec{\omega})$, where σ is a *surface measure* on S^2 that measures the area on the surface of the sphere. Notice that integration on $\mathbb{R}^3$ has different limits than integration on $\mathbb{R} \times S^2$.

Now, in $L^1(\mathbb{R}^2)$ the Radon Transform was defined using line integrals. We could certainly define a transform in $L^1(\mathbb{R}^3)$ using line integrals, but in the process we would lose the radial nature of the transform, and thus, the important central slice property. Moreover, since arbitrary lines in three-space are parameterized by four coordinates, the transform would not be very well-behaved, since it would take a system with three degrees of freedom to one with four, unless we restricted our attention to certain lines somehow.

Instead, the Radon Transform is generally defined in $\mathbb{R}^3$ using integrals along planes. This preserves the most important properties of the Radon Transform in $\mathbb{R}^2$. We begin by introducing our method of parameterizing planes in three-space:

Definition. For $r \in \mathbb{R}$ and $\vec{\omega} \in S^2$, let $l_{r,\vec{\omega}}$ be the 2D plane

$$\{\vec{x} \in \mathbb{R}^3 : \vec{x} \cdot \vec{\omega} = r\}.$$

In $\mathbb{R}^3$, the plane $l_{r,\vec{\omega}}$ is perpendicular to $\vec{\omega}$ and located r units from the origin.

Problem 1. Describe and draw the following planes:
(a) $l_{0,(1,0,0)}$ or $\langle \vec{x}, (1,0,0)\rangle = 0$,
(b) $l_{0,(0,-1,0)}$ or $\langle \vec{x}, (0,-1,0)\rangle = 0$,
(c) $l_{-2,(1,0,0)}$ or $\langle \vec{x}, (1,0,0)\rangle = -2$.

Problem 2. Write the plane passing through the following 3 points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ in the form $\langle \vec{x}, \vec{\omega}\rangle = t$ (i.e., find t and $\vec{\omega}$, and then find θ and φ such that $\vec{\omega} = \vec{\omega}(\theta, \varphi)$).

In order to define the 3D Radon transform, we need an interpretation for $\vec{\omega}^{\perp}$, since unlike the 2 dimensions there is no single perpendicular direction to $\vec{\omega}$, instead there is an entire plane perpendicular to $\vec{\omega}$. Thus, we choose *unit* vectors $\vec{u}_1$ and $\vec{u}_2$ such that

$$\langle \vec{\omega}, \vec{u}_1\rangle = 0, \quad \langle \vec{\omega}, \vec{u}_2\rangle = 0 \quad \text{and} \quad \langle \vec{u}_1, \vec{u}_2\rangle = 0.$$

Hence, the plane $W^{\perp} = span\,\{\vec{u}_1, \vec{u}_2\}$ is perpendicular to $\vec{\omega}$, and we have defined a new orthogonal coordinate system $U = \{\vec{\omega}, \vec{u}_1, \vec{u}_2\}$. This means that any $\vec{x} \in \mathbb{R}^3$ can be decomposed into $\vec{x} = \alpha_1\vec{\omega} + \alpha_2\vec{u}_1 + \alpha_3\vec{u}_2$ for some scalar coefficients $(\alpha_1, \alpha_2, \alpha_3)$; recall the notation from Linear Algebra Review $[\vec{x}]_U = (\alpha_1, \alpha_2, \alpha_3)^t$. In connection with the Radon Transform, we will write $\vec{x} = t\vec{\omega} + s_1\vec{u}_1 + s_2\vec{u}_2$, where $\langle \vec{x}, \vec{\omega}\rangle = t$.

Definition. The 3-dimensional *Radon transform* : $L^1(\mathbb{R}^3) \to L^1(\mathbb{R} \times S^2)$ is defined by

$$\mathcal{R}f(r, \vec{\omega}) = \int_{l_{r,\vec{\omega}}} f = \int_{\mathbb{R}^2} f(t\vec{\omega} + s_1\vec{u}_1 + s_2\vec{u}_2)\, ds_1 ds_2.$$

Definition. The 3-dimensional *Radon adjoint* (or the 3D back projection operator) maps functions on $\mathbb{R} \times S^2$ to functions on $\mathbb{R}^3$ and is defined for $h \in L^1(\mathbb{R} \times S^2) \cap L^2(\mathbb{R} \times S^2)$ by

$$\mathcal{R}^* h(\vec{x}) = \int_{S^2} h(\langle \vec{x}, \vec{\omega} \rangle, \vec{\omega})\, d\sigma(\vec{\omega}).$$

In the following two exercises use the explicit definition of the Radon transform and a change of variables $\vec{x} = t\vec{\omega} + s_1\vec{u}_1 + s_2\vec{u}_2$ to go from integrals in $\mathbb{R} \times S^2$ product to integrals in $\mathbb{R}^3$. It might be helpful to refer to the similar 2d proofs from class.

Problem 3. Prove the 3d Central Slice Theorem for $f \in L^1(\mathbb{R}^3)$:

$$\widetilde{\mathcal{R}f}(r, \vec{\omega}) \equiv \int_{-\infty}^{\infty} \mathcal{R}f(t, \vec{\omega})\, e^{-irt}\, dt = \hat{f}(r\vec{\omega}).$$

Problem 4. Show that the Radon adjoint on $\mathbb{R}^3$ is, in fact, the adjoint of the Radon Transform. For simplicity, assume that f is bounded and rapidly decreasing on $\mathbb{R}^3$ and h is bounded and rapidly decreasing in the affine parameter on $\mathbb{R} \times S^2$. Show that

$$\langle \mathcal{R}f, h \rangle_{L^2(\mathbb{R} \times S^2)} = \langle f, \mathcal{R}^* h \rangle_{L^2(\mathbb{R}^3)}.$$

We will examine one more property of the Radon transform, namely, the inversion of this transform, so we can reconstruct the function f from $\mathcal{R}f$. The Radon inversion formula in 3D is

$$f(\vec{x}) = \frac{1}{2} \frac{1}{(2\pi)^3} \int_{S^2} \int_{-\infty}^{\infty} \widetilde{\mathcal{R}f}(r, \vec{\omega})\, r^2\, e^{ir\langle \vec{x}, \vec{\omega} \rangle}\, dr\, d\sigma(\vec{\omega}).$$

In the class it was mentioned that the inversion of Radon transform can be done by using the '*Filter*' $\mathcal{GR}$ first and then the back projection which is the adjoint $\mathcal{R}^*$, i.e.,

$$f = \mathcal{R}^* \circ (\mathcal{GR}) f.$$

In 2D we will use *the Hilbert transform* to compute the Filter part (next week), but in 3D there is a simpler way to get the Filter $\mathcal{GR}$.

Problem 5. Show that the *Filter* part of the Radon inversion formula is an affine second derivative of the Radon transform:

$$(\mathcal{GR}) f(t, \vec{\omega})|_{t=\langle \vec{x}, \vec{\omega} \rangle} \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \widetilde{\mathcal{R}f}(r, \vec{\omega})\, r^2\, e^{ir\langle \vec{x}, \vec{\omega} \rangle}\, dr = -\frac{\partial^2}{\partial t^2} \mathcal{R}f(t, \vec{\omega})|_{t=\langle \vec{x}, \vec{\omega} \rangle}.$$

(*Work with the left hand side of the above statement, i.e., the middle expression : first, write out $\widetilde{\mathcal{R}f}$ using the definition (the affine 1d Fourier transform of $\mathcal{R}f$), then recall the property of the Fourier transform that a derivative becomes a multiplication by a variable on the Fourier side, i.e.,*

$$\widetilde{\partial_t f}(r, \vec{\omega}) = \int_{\mathbb{R}} \partial_t f(t, \vec{\omega}) e^{-itr}\, dt = -\int_{\mathbb{R}} f(t, \vec{\omega})\, (-ir)\, e^{-itr}\, dt = (ir) \widetilde{f}(r, \vec{\omega}).$$

Now find the same rule for the second partial derivative and use it in the written out middle expression.)