494/598 Fall 2008

ASSIGNMENT 11: RADON TRANSFORM

In this assignment we calculate several examples of Radon transforms.

1. Let $f(x,y) = \chi_{[-1,1]}(x)\left(\chi_{[-1,1]}(y) + \chi_{[2,4]}(y)\right)$.

a. Draw the contours of f (specifically, draw the jump).

<u>SOLUTION:</u>
The diagram should show two 2×2 squares, one centered at the origin and the other centered at (0,3).

b. Do the following:
(i) Find $\mathcal{R}f(t, \vec{\omega}\,)$ when $t > 1$ and $\vec{\omega} = (0,-1)$.
(ii) Find $\mathcal{R}f(t, \vec{\omega}\,)$ when $t < -1$ and $\vec{\omega} = (0,1)$.

<u>SOLUTION:</u>
By symmetry of the Radon Transform, these two must have the same value. In each case, the given t and $\vec{\omega}$ define lines which lie entirely outside the support of f, so the correct answer is 0.

c. Find:
(i) $\mathcal{R}f(0,(1,0))$,
(ii) $\mathcal{R}f(0,(0,1))$,
(iii) $\mathcal{R}f(0,(\cos(\pi/6),\sin(\pi/6)))$,
(iv) $\mathcal{R}f(\sqrt{2}/2,(\sqrt{2}/2,\sqrt{2}/2))$.

<u>SOLUTION:</u>

(i) $\mathcal{R}f(0,(1,0)) = 4$, because the line is a vertical line which passes through both boxes.

(ii) $\mathcal{R}f(0,(0,1)) = 2$, because the line is horizontal, passing through only one box.

(iii) $\mathcal{R}f(0,(\cos(\pi/6),\sin(\pi/6))) = 4/\sqrt{3}$, because the given line is oriented 30 degrees off of horizontal, and so passes through only one box and length of the hypotenuse of a 30/60/90 triangle with long side equal to 1 is $2/\sqrt{3}$.

(iv) $\mathcal{R}f(\sqrt{2}/2,(\sqrt{2}/2,\sqrt{2}/2)) = 2\sqrt{2}$. The reasoning is as in (iii), except that this is a 45/45/90 triangle.

2. Let $f(x,y) = e^{-(1+|x|)(1+|y|)}$. Find $\mathcal{R}f(t,(0,1))$ for each t. In other words, express it in terms of t.

<u>SOLUTION:</u>
Let

$$f(x,y) = e^{-(1+|x|)(1+|y|)}.$$

We have that $\vec{\omega} = (0,1)$, so that $\vec{\omega}^{\perp} = (-1,0)$. In this case we can equivalently use (1,0) in the parametrization, which makes the integration slightly easier. This is possible because if we make a substitution $s' = -s$, then what will be exactly what we get. Using this observation, and symmetry of the integral, we get

$$\begin{aligned}
\mathcal{R}f(t,(0,1)) &= \int_{-\infty}^{\infty} f(t(0,1)+s(1,0))\,ds \\
&= \int_{-\infty}^{\infty} f(s,t)\,ds \\
&= \int_{-\infty}^{\infty} e^{-(1+|s|)(1+|t|)}\,ds = \int_{-\infty}^{0} \ldots + \int_{0}^{\infty} \\
&= 2\int_{0}^{\infty} e^{-(1+s)(1+|t|)}\,ds \\
&= 2\left[-\frac{1}{1+|t|}e^{-(1+s)(1+|t|)}\right]_0^{\infty} \\
&= 2\,\frac{e^{-(1+|t|)}}{1+|t|}.
\end{aligned}$$

3. For $\vec{x} = (x_1, x_2)$ define

$$f(\vec{x}) = \begin{cases} 3x_1^2 + x_2^2 + 15\,x_1x_2 & \text{for} \quad |x_1| < 1,\ |x_2| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find $\mathcal{R}f(0,(\cos(2\pi/3),\sin(2\pi/3)))$.

Solution: Let

$$f(\vec{x}) = \begin{cases} 3x_1^2 + x_2^2 + 15x_1x_2, & |x_1| < 1 \text{ and } |x_2| < 1 \\ 0. & \text{otherwise.} \end{cases}$$

We want to find $\mathcal{R}f(0,(\cos(\frac{2\pi}{3}),\sin(\frac{2\pi}{3})))$. We have $\vec{\omega} = (-\frac{1}{2},\frac{\sqrt{3}}{2})$, and thus, $\vec{\omega}^{\perp} = (-\frac{\sqrt{3}}{2},-\frac{1}{2})$. Drawing a diagram and using the same trigonometric ideas as in part (ii) of (1c), we find that the limits of integration in s should be $-2/\sqrt{3}$ and $2/\sqrt{3}$. We will, as in the last problem, use $-\vec{\omega}^{\perp}$ in this problem in order to make the algebra as easy as possible. We have

$$\begin{aligned}
\mathcal{R}f\left(0,\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)\right) &= \int_{-\infty}^{\infty} f\left((0,0)+\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)s\right)ds \\
&= \int_{-2/\sqrt{3}}^{2/\sqrt{3}} \left[3\left(\frac{\sqrt{3}}{2}s\right)^2 + \left(\frac{1}{2}s\right)^2 + 15\left(\frac{\sqrt{3}}{2}s\right)\left(\frac{1}{2}s\right)\right] ds \\
&= \int_{-2/\sqrt{3}}^{2/\sqrt{3}} \left[\left(\frac{5}{2}+\frac{15\sqrt{3}}{4}\right)s^2\right] ds \\
&= \left[\frac{1}{3}\left(\frac{5}{2}+\frac{15\sqrt{3}}{4}\right)s^3\right]_{-2/\sqrt{3}}^{2/\sqrt{3}} \\
&= \frac{4}{3\sqrt{3}}\left(\frac{5}{2}+\frac{15\sqrt{3}}{4}\right) = \frac{1}{3\sqrt{3}}(10+45) \\
&= \frac{55}{3\sqrt{3}}.
\end{aligned}$$

4. Let $f(x,y) = (x^2+y^2)\left(\chi_{[-1,1]}(x)\,\chi_{[-1,1]}(y)\right)$. Find $\mathcal{R}f(\sqrt{2}/2, (\sqrt{2}/2, \sqrt{2}/2))$.

SOLUTION:

Let $f(x,y) = (x^2+y^2)\chi_{[-1,1]}(x)\chi_{[-1,1]}(y)$. We want to find $\mathcal{R}f(\frac{\sqrt{2}}{2}, (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}))$. Here $\vec{\omega}^{\perp} = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, and drawing a careful diagram, we see using elementary trigonometry that the limits of integration should be $-\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{2}}{2}$. Moreover, $t\vec{\omega} = (\frac{1}{2}, \frac{1}{2})$. We find that

$$\begin{aligned}
\mathcal{R}f\left(\frac{\sqrt{2}}{2}, \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\right) &= \int_{-\infty}^{\infty} f\left((\frac{1}{2}, \frac{1}{2}) + \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) s\right) ds \\
&= \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \left[\left(\frac{1}{2} - \frac{\sqrt{2}}{2}s\right)^2 + \left(\frac{1}{2} + \frac{\sqrt{2}}{2}s\right)^2\right] ds \\
&= \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \left(\frac{1}{2} + s^2\right) ds \\
&= \left[\frac{1}{2}s + \frac{1}{3}s^3\right]_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \\
&= \left(\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{12}\right) - \left(-\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{12}\right) \\
&= \frac{\sqrt{2}}{2}
\end{aligned}$$

5. Let $f(x,y) = e^{-\pi(x^2+y^2)}$. Find $\mathcal{R}f(t, \vec{\omega})$.

SOLUTION:

Since f is circularly symmetric, we may without loss of generality assume $\vec{\omega} = (0,1)$. In this case, we have

$$\mathcal{R}f(t,\vec{\omega}) = \int_{-\infty}^{\infty} e^{-\pi(s^2+t^2)} ds = e^{-\pi t^2} \int_{-\infty}^{\infty} e^{-\pi s^2} ds.$$

Although there is no closed form antiderivative for the Gaussian, we know from our work in the the assignment on the Fourier transform that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, so that

$$\int_{-\infty}^{\infty} e^{-\pi s^2} ds = \int_{-\infty}^{\infty} e^{-(\sqrt{\pi}s)^2} ds = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} ds = 1.$$

It follows that

$$\mathcal{R}f(t,\vec{\omega}) = e^{-\pi t^2}.$$