494/598 Fall 2008

ASSIGNMENT 11: RADON TRANSFORM

In this assignment, we calculate several examples of Radon transforms. Recall the definition for the Radon transform:

$$\mathcal{R}f(t,\vec{\omega}) = \int_{l_{t,\vec{\omega}}} f\,ds,$$

where the line $l_{t,\vec{\omega}} = \{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 : \langle \vec{x}, \omega \rangle = t\}$ is parameterized by $\{s \in (-\infty, \infty) : \boldsymbol{x} = t\vec{\omega} + s\,\vec{\omega}^{\perp}\}$. So in the the above integral $f = f(\boldsymbol{x}) = f(t\vec{\omega} + s\,\vec{\omega}^{\perp})$ and the integral can be rewritten as

$$\mathcal{R}f(t,\vec{\omega}) = \int_{\infty}^{\infty} f(t\vec{\omega} + s\,\vec{\omega}^{\perp})\,ds.$$

1. Let $f(x,y) = \chi_{[-1,1]}(x)\left(\chi_{[-1,1]}(y) + \chi_{[2,4]}(y)\right)$.

a. Draw the contours of f (specifically, draw the jump).

b. Do the following:

(i) Find $\mathcal{R}f(t,\vec{\omega}\,)$ when $t > 1$ and $\omega = (0,-1)$.

(ii) Find $\mathcal{R}f(t,\vec{\omega}\,)$ when $t < -1$ and $\omega = (0,1)$.

c. Find:

(i) $\mathcal{R}f(0,(1,0))$,

(ii) $\mathcal{R}f(0,(0,1))$,

(iii) $\mathcal{R}f(0,(\cos(\pi/6),\sin(\pi/6)))$,

(iv) $\mathcal{R}f(\sqrt{2}/2,(\sqrt{2}/2,\sqrt{2}/2))$.

2. Let $f(x,y) = e^{-(1+|x|)(1+|y|)}$. Find $\mathcal{R}f(t,(0,1))$ for each t. In other words, express it in terms of t.

(*It may be helpful to replace $\omega^{\perp}$ with $-\omega^{\perp}$ in the parametrization, since then the parametrization will move in the positive x direction.*)

3. For $\vec{x} = (x_1, x_2)$ define

$$f(\vec{x}) = \begin{cases} 3x_1^2 + x_2^2 + 15\,x_1x_2 & \text{for} \quad |x_1| < 1,\ |x_2| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find $\mathcal{R}f(0,(\cos(2\pi/3),\sin(2\pi/3)))$.

(*It may be helpful, as in the last exercise, to use $-\omega$ in the parametrization; this simplifies the algebra.*)

4. Let $f(x,y) = (x^2+y^2)\left(\chi_{[-1,1]}(x)\,\chi_{[-1,1]}(y)\right)$. Find $\mathcal{R}f(\sqrt{2}/2,(\sqrt{2}/2,\sqrt{2}/2))$.

(*Be careful with this parametrization, because the origin does not lie on the line along which we are integrating.*)

5. Let $f(x,y) = e^{-\pi(x^2+y^2)}$. Find $\mathcal{R}f(t,\vec{\omega})$.