ASSIGNMENT 1

Exercise 1. (The Unit Ball) Recall the definitions of the sup norm $\|\cdot\|_{max}$, the Euclidean norm $\|\cdot\|_2$, the additive norm $\|\cdot\|_1$ and the p-norms $\|\cdot\|_p$ on $\mathbb{R}^n$. Draw diagrams of the closed unit ball in $\mathbb{R}^2$, $B_1 = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$, under each of these norms. (Draw a separate diagram for $p > 1$ and $p < 1$ cases, note that for $p < 1$ $\|\cdot\|_p$ generates only a semi-norm (what fails?).)

Exercise 2. (Euler's formula) Let $x \in \mathbb{R}$. Show that $e^{ix} = \cos x + i \sin x$.

Hint: Use the definition of the complex exponential from the Lecture Notes, i.e., the Taylor expansion at 0. The proof requires a rearranging of terms, but since the series is absolutely convergent we can make such a rearrangement.

Exercise 3. (Square) Show that for $z \in \mathbb{C}$, $z\bar{z} = |z|^2$. Do this using both the rectangular and polar representations of z (it should only take a couple of lines both ways). Is z^2 different from $|z|^2$, or $|z^2|$?

Exercise 4. (Linear Dependence and Bases) Prove statements, following the given steps:

(a) If $n+1$ vectors u_j (i.e., $u_1, u_2, ..., u_{n+1}$) lie in the span of n vectors v_l (i.e., $v_1, v_2, ..., v_n$), then the $n+1$ vectors u_j's are linearly dependent.

Sketch: Proceed by induction:

- *First, write out the problem for* $n = 1$ *and see that it is true. (You might also want to check* $n = 2$ *to understand the set up better.)*
- *For the inductive step, assume the conclusion when* $n = k - 1$ *(write it out).*
- *Now we will take* $n = k$ *and the goal will be to show that* $u_1, u_2, ..., u_{k+1}$ *are linearly dependent. Write each* u_j *in terms of the* v_l*'s* $(l = 1, ..., k)$*. Assume that the coefficient of* v_k *in the expansion of* u_{k+1} *is nonzero (can you justify this assumption? remember how Steven discussed on the board different cases why the coefficient in front of* v_k *can be chosen nonzero?). Using this, subtract a multiple of* u_{k+1} *from* u_j *to create a new set of* k *vectors* w_j *such that each* w_j *lies in the span of* $\{v_1, ..., v_{k-1}\}$*—note that in order for this to occur, the coefficients of* v_k *in the expansions need to cancel out.*
- *Using the inductive hypothesis, conclude that the* w_j*'s are linearly dependent (there are only* k *of them!). Finally, show that this implies that the* u_j*'s are linearly dependent (this again requires expanding in terms of the* v_l*'s).*

(b) Let W be a subspace of a vector space V over $\mathbb{F}$. Let B_1 and B_2 be bases for W, and suppose that B_1 contains n elements. Then B_2 also contains n elements.

(*Hint: Use part (a).*)

(c) Let V be an n-dimensional vector space, and $v_1, ..., v_n$ be distinct vectors in V. Then $v_1, ..., v_n$ are linearly independent if and only if they span V.

(*Hint: Use part (a) for both implications.*)

Exercise 5. (Inner Product Space $\longrightarrow$ Normed Space $\longrightarrow$ Metric Space) Here we work in the opposite direction as the lecture notes on metrics, norms and inner products. Using the suggestions that follow, prove that **an inner product space is a metric space** by first proving the Cauchy-Schwartz inequality, and then showing that an inner product induces a norm and a norm induces a metric:

(a) (Cauchy-Schwartz Inequality) Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space and let $\|\cdot\| : V \to \mathbb{R}$ be a norm induced by the inner product (recall what this means). Show that for any $x, y \in V$,

$$|\langle x, y\rangle| \leq \|x\| \, \|y\| \, .$$

Follow the steps below.

(i) First, suppose $y = 0$. Show that the inequality holds. For the remainder of the proof, we will assume $y \neq 0$.

(ii) For $x,\ y \in V$, define $F_{x,y} : \mathbb{C} \to \mathbb{C}$ by $F_{x,y}(t) = \langle x + ty, x + ty\rangle$. Expand F using linearity and complex linearity of the inner product. What inequality do we know $F_{x,y}$ satisfies?

(iii) Now—just for this part—assume we are dealing with a real inner product and $F_{x,y} : \mathbb{R} \to \mathbb{R}$. Use elementary calculus to find the only critical point of $F_{x,y}$,, that is, the point at which $F_{x,y}$ takes its minimum. [The solution is the minimum even when the inner product is complex, but it is not obvious that we can use methods of elementary calculus with a function of a complex variable.] Explain why the nonzero assumption on y is necessary.

(iv) Next we need the following: if $z \in \mathbb{C}$, then $z\bar{z} = |z|^2$.

(v) Now use the inequality from part (ii), the solution found in part (iii) [Remark: we found it using calculus with the assumption that the inner product was real, but it works for the complex case as well], and the lemma in part (iv) to prove the Cauchy-Schwartz inequality.

(b) Prove that $\|\cdot\|$ induced by an inner product $< \cdot, \cdot >$ is indeed a norm on V.

Hint: positive homogeneity and positive definiteness are easy. For the triangle inequality, try squaring both sides.

(c) Let $(V, \|\cdot\|)$ be a normed space. Define $d : V \times V \to \mathbb{R}$ by $d(x, y) = \|x - y\|$. Prove that (V, d) is a metric space.