Math 598, Spring 2006

HOMEWORK 3

(due Tuesday, March 21)

1. Recall that in the theorem on approximation of any L^p function by $C^\infty_{cmpt}(\Omega)$ sequence, we constructed a function $\Phi_K = \phi_d * \chi_{K_+}$, where
- $\phi \in C^\infty$ with $\operatorname{supp}\phi \subseteq B(0,1)$, $0 \le \phi(x) \le 1$, $\int \phi(x)\,dx = 1$ and $\phi_\epsilon = \frac{1}{\epsilon^n}\phi(\frac{x}{\epsilon})$;
- K is compact in Ω, an open set in $\mathbb{R}^n$;
- $K_+ = \{x : |x-y| \le d \text{ for some } y \in K\}$, note $K \subset K_+ \subset \Omega$ and
- d was chosen such that "$2d$"-annulus around K is in Ω,
i.e., there exists d such that $\{x : |x-y| \le 2d\} \subset \Omega$.

Check that Φ_K satisfies the following conditions:
(i) $\Phi_K \in C^\infty_{cmpt}(\Omega)$
(ii) $\Phi_K = \begin{cases} 1 & x \in K, \\ \text{between 0 and 1} & x \in \Omega \setminus K \\ 0 & x \in \Omega^c \end{cases}$

2. Prove Riemann-Lebesgue property:
$\hat{f}(\xi) \to 0$ when $|\xi| \to \infty$.

3. Consider the addition and multiplication operations:
there is always an element g such that adding/multiplying it to a scalar (function) doesn't change that scalar (function), namely,
- for addition $g = 0$, i.e., $0 + f = f$;
- for multiplication $g = 1$, i.e., $1 \cdot f = f$.

One would like to have the same property with convolution, i.e.
for some g, $g * f = f$.

Show that for $f \in L^1(\mathbb{R})$, then such an element g does not exist in L^1.
What about other L^p?