Math 598, Spring 2006

HOMEWORK 1

(due Tuesday, February 14)

1. Consider $L^p(\mathbb{R})$, $1 \le p < \infty$.
Show that the sequence $\{f_k\}_k$ in $L^p(\mathbb{R})$ converges weakly to zero but does not converge strongly to zero (or anything else):

(i) $f_k(x) = \begin{cases} \sin kx & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

(ii) $f_k(x) = k^{1/p} g(kx)$, where $g \in L^1(\mathbb{R})$ is any fixed function

(iii) $f_k(x) = g(x+k)$, where $g \in L^1(\mathbb{R})$ is any fixed function

2. Let $f,\, g \in L^1(\mathbb{R}^n)$. Prove that the convolution $f * g$, defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy,$$

is also in $L^1(\mathbb{R}^n)$ and $\|f * g\|_{L^1} \le c\,\|f\|_{L^1}\|g\|_{L^1}$.

3. (BONUS) Prove the generalization of the previous problem:
Given $f \in L^p$ and $g \in L^q$, show that

$$\|f * g\|_{L^r} \le c\,\|f\|_{L^p}\|g\|_{L^q},$$

provided $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$.