LECTURE NOTES FOR MATH 494, SPRING 2005

2. **Fourier Transform**

1. FIRST ATTEMPT TO FOURIER TRANSFORM.

Consider the Fourier Series on $[-\pi, \pi)$ or $[a, a+2\pi)$, $a \in \mathbb{R}$. Any $f \in L^2(-\pi, \pi)$ can be represented in terms of the Fourier Series

$$f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}, \tag{1}$$

where

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx}\, dx = \left\langle f, e^{inx} \right\rangle. \tag{2}$$

This comes from the fact that $\{\frac{1}{\sqrt{2\pi}} e^{inx}\}_{n\in\mathbb{Z}}$ is an o.n. sequence in $L^2(-\pi, \pi)$. The completeness of this sequence comes from the density of trigonometric polynomials in $L^2(-\pi, \pi)$, refer to the textbook.

Our goal is to consider functions on $L^2(\mathbb{R})$ and obtain something similar to (1) and (2) for such functions, i.e.,

$$f(x) \;\sim\; \int \hat{f}(\xi)\, e^{ix\xi}\, d\xi$$

with

$$\hat{f}(x) \;\sim\; \int f(x) e^{-ix\xi}\, dx.$$

The obstacles arising right away are

- $\{e^{ix\xi}\}$ can not be an o.n. set in $L^2(\mathbb{R})$, since the set is uncountable, moreover, $e^{ix\xi} \notin L^2(\mathbb{R})$
- the integral $\displaystyle\int f(x) e^{-ix\xi}\, dx$ does not converge absolutely for $f \in L^2(\mathbb{R})$

So instead we define the Fourier Transform for integrable functions only (i.e. on L^1):

Definition 2.1. *For $f \in L^1(\mathbb{R})$ define $\hat{f} : \mathbb{R} \to \mathbb{C}$ by*

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx.$$

Note that here the integral converges absolutely.

2. CONVOLUTION.

Consider $f, g : \mathbb{R} \to \mathbb{C}$ and suppose that

$$\int_{\mathbb{R}} f(y) g(x-y)\, dy < \infty \quad \text{for a.e. } x \in \mathbb{R}.$$

Define $f * g$, the convolution of f and g as $f * g = \displaystyle\int_{\mathbb{R}} f(y) g(x-y)\, dy$ (and $f * g = 0$ on the set where the integral is infinite).

3. PROPERTIES OF CONVOLUTION.

(a) $f * g = g * f$ (this comes from the change of variables)

(b) $|f * g| \le \|f\|_{L^2} \|g\|_{L^2}$ (apply Cauchy-Schwarz)

(c) $\|f * g\|_{L^1} \le \|f\|_{L^1} \|g\|_{L^1}$, provided $f, g \in L^1$.

To prove this, note

$$\int |(f*g)(x)|\, dx \le \int \int |f(x-y)|\, |g(y)|\, dx\, dy \le \left(\int |f(z)|\, dz \right) \left(\int |g(y)|\, dy \right),$$

where the first inequality uses Fubini's Theorem and second change of variables $z = x - y$.

(d) $\|f * g\|_{L^2} \le \|f\|_{L^2}\|g\|_{L^1}$, provided $f \in L^2$ and $g \in L^1$.
To prove we need the integral version of Minkowski's inequality (see Lecture 1) (which is just a triangle inequality for norms, not in a discrete sum but in continuous way - integral):

$$\left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x-y)g(y)\,dy \right|^2 dx\right)^{1/2} \le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x-y)|^2 |g(y)|^2\,dx \right)^{1/2} dy.$$

Remark: This inequality can be generalized to $f \in L^p$, $1 \le p < \infty$.

(e) $(f * g)\hat{} = \hat{f} \cdot \hat{g}$, for $f, g \in L^1(\mathbb{R})$
Switch the order of integration (Fubini's Theorem): $(f * g)\hat{}(\xi) = \int\int f(x - y)g(y)e^{-ix\xi}\,dy\,dx = \int g(y)e^{-iy\xi}\left(\int f(x-y)e^{-i(x-y)\xi}\,dx\right)dx = \hat{f}(\xi)\hat{g}(\xi)$.

The next property gives an approximation of an L^1 function by convolution with an "identity". First we need definitions.

4. APPROXIMATE IDENTITY. Suppose a function g has the following properties:

- $\int_{\mathbb{R}} g(x)\,dx = 1$,
- $|g(x)| \le \dfrac{c}{(1+|x|)^2}$.

Note that $g \in L^1(\mathbb{R})$.

Denote $g_t(x) = \frac{1}{t} g\left(\frac{x}{t}\right)$ for $t > 0$. Observe that $\hat{g}_t(\xi) = \hat{g}(t\xi)$.
Also note that $\int_{\mathbb{R}} g_t(x)\,dx = \int_{\mathbb{R}} g(u)\,du = 1$.

Definition 2.2. *The family $\{g_t\}_{t>0}$ is called an approximate identity if g satisfies the above two properties.*

5. APPROXIMATION OF AN L^1 FUNCTION BY CONVOLUTION.

Lemma 2.3. *For $f \in L^1$ and $\{g_t\}$ an approximate identity, we have*

(3) $$\lim_{t\to 0^+} (g_t * f)(x) = f(x) \quad a.e.\ on\ \mathbb{R}.$$

In order to prove this Lemma we need a Lebesque point notation:

Definition 2.4. *The point x is a Lebesque point of function f if*

$$\lim_{h\to 0^+} \frac{1}{2h}\int_{-h}^{h} |f(x-y) - f(x)|\,dy = 0$$

Remark. If $f \in L^1(\mathbb{R})$, then almost every point is a Lebesque point of f. (HW Ex 5.1.11).

Proof. Fix a Lebesque point x and $\epsilon > 0$. Then there exists H such that for $0 < h < H$ we have

(4) $$\frac{1}{2h}\int_{-h}^{h} |f(x-y) - f(x)|\,dy < \tilde{\epsilon}$$

(not to get very technical we will specify $\tilde{\epsilon}$ in terms of ϵ at the end of the proof).

We will show that at every Lebesque point x of f we have $(g_t * f)(x) \longrightarrow f(x)$ as $t \to 0$, i.e.

$$\int_{\mathbb{R}} |f(x-y) - f(x)|\,|g_t(y)|\,dy \to 0.$$

Split this integral into two: $\int_{|y|<H}$ and $\int_{|y|\geq H}$. Then for the second integral (when y is large) we use

$$|g_t(y)| \leq \frac{c}{t(1+\frac{|y|}{t})^2} \leq \frac{c\,t}{|y|^2} \leq \frac{c\,t}{H^2}.$$

Thus,

$$\int_{|y|\geq H} |f(x-y)-f(x)|\,|g_t(y)|\,dy \leq \int_{|y|\geq H} |f(x-y)|\,|g_t(y)|\,dy$$

$$+|f(x)|\int_{|y|\geq H} |g_t(y)|\,dy \leq \frac{c\,t}{H^2}\|f\|_{L^1(\mathbb{R})} + |f(x)|\int_{|y|\geq H} |g_t(y)|\,dy < \frac{\epsilon}{4}+\frac{\epsilon}{4} < \frac{\epsilon}{2},$$

since the first term can be made as small as possible as $t \to 0$ and the second term decreases to 0, since the area of the "tail" of $g_t(x)$ vanishes as $t \to 0$ (the bulk of $g_t(x)$ is inside of $|y| < H$).

Now for the first term $\int_{|y|\geq H}$ we can assume that $t < H$, since $t \to 0$. Choose an integer M such that $\frac{H}{2^{M+1}} < t \leq \frac{H}{2^M}$. Consider $\frac{H}{2} < |y| \leq H$. Then

$$|g_t(y)| \leq \frac{c}{t(1+\frac{H}{2t})^2} \leq \frac{2tc}{H^2}.$$

In general, for $\frac{H}{2^{k+1}} < |y| \leq \frac{H}{2^k}$ we have $|g_t(y)| \leq \frac{2^k tc}{H^2}$. For the "last" interval $|y| \leq \frac{H}{2^{M+1}}$ we get $|g_t(y)| \leq \frac{c}{t}$.

Now, subdivide the integral $\displaystyle\int_{|y|<H}$ into $\displaystyle\sum_{k=1}^{M+1}\int_{\frac{H}{2^{k+1}}<|y|\leq\frac{H}{2^k}} + \int_{|y|\leq\frac{H}{2^{M+1}}}$.

Then

$$\int_{|y|<H} |f(x-y)-f(x)|\,|g_t(y)|\,dy \leq \sum_{k=1}^{M+1}\frac{2^k tc}{H^2}\int_{\frac{H}{2^{k+1}}<|y|\leq\frac{H}{2^k}} |f(x-y)-f(x)|\,dy$$

$$+\frac{c}{t}\int_{|y|\leq\frac{H}{2^{M+1}}} |f(x-y)-f(x)|\,dy.$$

Using the Lebeque point estimate (4), we obtain

$$\int_{\frac{H}{2^{k+1}}<|y|\leq\frac{H}{2^k}} |f(x-y)-f(x)|\,dy \leq \frac{2H\tilde{\epsilon}}{2^k}$$

and so we get the following bound on the first integral

$$\int_{|y|<H} \dots \leq \sum_{k=1}^{M+1}\frac{2^k tc}{H^2}\frac{2H\tilde{\epsilon}}{2^k} + \frac{c}{t}\frac{2H\tilde{\epsilon}}{2^{M+1}} \leq \frac{ct\tilde{\epsilon}}{H}\sum_{k=1}^{M+1}2^k + 2c\tilde{\epsilon}$$

$$\leq \frac{ct\tilde{\epsilon}}{H}4\cdot 2^M + 2c\tilde{\epsilon} \leq 6c\tilde{\epsilon} < \epsilon/2,$$

where we used $\dfrac{H}{t2^{M+1}} \leq 1$ and $\dfrac{t2^M}{H} \leq 1$ by the choice of M (note that we need to choose $\tilde{\epsilon} < \frac{\epsilon}{12c}$).

Combining both estimates, we obtain

$$\int_{\mathbb{R}} |f(x-y)-f(x)|\,|g_t(y)|\,dy < \epsilon.$$

$\square$

6. GAUSSIAN DISTRIBUTION.

An example of an approximate identity is *Gaussian distribution* which is defined as

$$G(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

It is a bell shaped curve with maximum at $x = 0$ and decaying at infinity.

Properties of $G(x)$.

- $\int_{\infty}^{\infty} G(x)\,dx = 1$ (in order to calculate this (Ex 5.2.2) multiply two integrals in x and y variables, then switch to polar coordinates; since $dx\,dy = r\,dr\,d\theta$, change of variables will give the answer)
- For each $N > 0$ there exists a constant $c_N > 0$ such that

  $$|G(x)| \le \frac{c_N}{(1+|x|)^N}$$

  (exponential decay is faster than any power N).
- $\hat{G}(\xi) = e^{-x^2/2}$ (or $\hat{G} = \sqrt{2\pi}G$).

  To prove this, consider the ODE

  $$G'(x) + xG(x) = \frac{-x}{\sqrt{2\pi}} e^{-x^2/2} + x\frac{e^{-x^2/2}}{\sqrt{2\pi}} = 0.$$

  Taking the Fourier transform, we get

  $$i\xi\hat{G}(\xi) + i\hat{G}'(\xi) = 0,$$

  which is

  $$\frac{d\hat{G}}{G} = -\xi\,d\xi,$$

  solution of which is

  $$\hat{G}(\xi) = \hat{G}(0)e^{-\xi^2/2}, \quad \text{where} \quad \hat{G}(0) = \int_{\infty}^{\infty} G(x)\,dx = 1.$$

Define $G_t(x) = \frac{1}{t}G\left(\frac{x}{t}\right)$. The family $\{G_t\}_{t>0}$ is an approximate identity and $G_t * f \to f$ a.e. as $t \to 0$. Note that $\int_{\mathbb{R}} G_t(x)\,dx = \int_{\mathbb{R}} G(x)\,dx = 1$.

7. FOURIER INVERSION.

Definition 2.5. *For $g \in L^1(\mathbb{R})$ define*

$$g^{\vee}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} g(\xi)e^{ix\xi}\,d\xi.$$

Note that the integral converges absolutely.

Hope: $(\hat{f})^{\vee} = f$ (so we would have the chain $f \xrightarrow{\wedge} \hat{f} \xrightarrow{\vee} f$), and thus,

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)e^{ix\xi}\,d\xi \tag{5}$$

Theorem 2.6. *Suppose that both $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$. Then (5) holds a.e. on $\mathbb{R}$.*

Proof. By the property of an approximate identity using the gaussian distribution, we have

$$\lim_{t\to 0^+}(G_t * f)(x) = f(x) \quad \text{a.e.}$$

Consider the convolution term $G_t(x-y)$ more carefully:

$$G_t(x-y) = \frac{1}{t}G\left(\frac{x-y}{t}\right) = \frac{1}{t\sqrt{2\pi}}\hat{G}\left(\frac{x-y}{t}\right) = \frac{1}{t\sqrt{2\pi}}\hat{G}\left(\frac{y-x}{t}\right)$$

$$= \frac{1}{t\sqrt{2\pi}}\int_{\mathbb{R}} G(s)e^{-i\frac{y-x}{t}s}\,ds \int_{\mathbb{R}} \hat{G}(\xi t)e^{-i(y-x)\xi}\,d\xi,$$

where we used properties of G, also G being even and change of variables $s = \xi t$. Then the convolution between G_t and f can be expressed as

$$\int_{\mathbb{R}} f(y)G_t(x-y)\,dy = \int_{\mathbb{R}} f(y)\left[\int_{\mathbb{R}} \hat{G}(\xi t)e^{-iy\xi}e^{ix\xi}\,d\xi\right]dy$$

$$= \int_{\mathbb{R}} \hat{G}(\xi t)e^{ix\xi}\left[\int_{\mathbb{R}} f(y)e^{-iy\xi}\,dy\right]d\xi = \int_{\mathbb{R}} \hat{f}(\xi)\hat{G}(\xi t)e^{ix\xi}\,d\xi \longrightarrow \int_{\mathbb{R}} f(\xi)e^{ix\xi}\,d\xi,$$

by using Fubini's theorem to switch the order of integration and Lebesque dominated convergence theorem (LDCThm) to bring the limit on t inside the integral as well as $\hat{G}(t\xi) = e^{-\frac{\xi^2 t^2}{2}} \to 1$ as $t \to 0$. □

Remark. By LDCThm we have $\lim_{t\to 0}\hat{G}_t(\xi)\hat{t}(\xi) = \hat{t}(\xi)$ which suggests looking for a function g such that $\hat{g}(\xi) = \lim_{t\to 0}\hat{G}_t(\xi) = 1$, i.e. can we have a function g such that $g = 1^{\vee}$? Write $g(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\,d\xi$, examine it and observe that it does not converge. In the theory of distribution it is possible to define a "delta function" δ which is a distribution such that $\hat{\delta} = 1$. (See later lectures for distributions.)

8. Uniqueness of the Fourier Transform on $L^1(\mathbb{R})$.

Suppose that there exist $f, g \in L^1(\mathbb{R})$ and such that $\hat{f} = \hat{g}$ a.e. Define $F = f - g$. Then $\hat{F} = \hat{f} - \hat{g} = 0$, and so, $F^{\vee} = 0^{\vee} = 0$ a.e. Thus, the Fourier Transform is unique on $L^1(\mathbb{R})$.

9. Connection with $L^2(\mathbb{R})$.

Observation #1. If $f, \hat{f} \in L^1(\mathbb{R})$, then $f, \hat{f} \in L^2(\mathbb{R})$.

Proof. $\int_{\mathbb{R}} |\hat{f}(\xi)|^2\,d\xi = \int_{\mathbb{R}} \hat{f}(\xi)\,\overline{\hat{f}(\xi)}\,d\xi \le \sup_{\xi\in\mathbb{R}} |\hat{f}(\xi)| \int_{\mathbb{R}} |\overline{\hat{f}(\xi)}|\,d\xi \le \|f\|_{L^1(\mathbb{R})} \cdot \|\hat{f}\|_{L^1(\mathbb{R})}$, by using the first property of the Fourier transform. Repeat the same calculation for f and use inverse Fourier transform properties. □

Observation #2. $\left\langle \hat{f}, \hat{g} \right\rangle = 2\pi \langle f, g\rangle$.

Proof. $\int_{\mathbb{R}} \hat{f}(\xi)\,\overline{\hat{g}(\xi)}\,d\xi = \int_{\mathbb{R}} \hat{f}(\xi)\int_{\mathbb{R}} \bar{g}(\xi)e^{ix\xi}\,dx\,d\xi = \int_{\mathbb{R}} \bar{g}(\xi)\int_{\mathbb{R}} \hat{f}(\xi)e^{ix\xi}\,d\xi\,dx = 2\pi\langle f, g\rangle$, where we used Fubini's theorem to switch the order of integration. □

Observation #3. $\|\hat{f}\|_{L^2(\mathbb{R})} = 2\pi\|f\|_{L^2(\mathbb{R})}$ (apply above to $g = f$).

Observation #4. If $f \in L^1(\mathbb{R})$, then $\hat{f}$ is bounded and continuous on $\mathbb{R}$; moreover, if $\hat{f} \in L^1(\mathbb{R})$, then $(\hat{f})^{\vee}$ is bounded and continuous. Thus, if both f and $\hat{f} \in L^1(\mathbb{R})$, then $f = (\hat{f})^{\vee}$ a.e. and f is bounded and continuous a.e. on $\mathbb{R}$.

Question: Can we have a function such that $f, \hat{f} \in L^1(\mathbb{R})$ at the same time?

Observation #5. Let $f \in \mathbb{C}_c^2(\mathbb{R})$, i.e. twice continuously differentiable and has a compact support. Then obviously $f \in L^1(\mathbb{R})$, $\hat{f}$ is bounded and using integration by parts $|\hat{f}(\xi)| \leq \left|\frac{1}{i\xi}\int_{\mathbb{R}} f'(x)e^{-ix\xi}\,dx\right| \leq \left|\frac{1}{(i\xi)^2}\int_{\mathbb{R}} f''(x)e^{-ix\xi}\,dx\right| \leq \frac{c}{|\xi|^2}$ (boundary values vanish because of the compact support of f). Hence, $\hat{f} \in L^1(\mathbb{R})$.
By observation #1 it follows that $f, \hat{f} \in L^2(\mathbb{R})$!

Observation #6. Suppose $f \in L^2(\mathbb{R})$. We want to choose a sequence $\{f_n\} \in L^1(\mathbb{R})$ such that $f_n \to f$ in the L^2 norm and $\hat{f}_n \in L^1(\mathbb{R})$. Then our goal would be to consider $\lim_{n\to\infty} \hat{f}_n$ and possibly define the Fourier transform of an L^2 function.

Proposition 2.7. *Any $f \in L^2(\mathbb{R})$ can be approximated by a sequence of functions from $\mathbb{C}_c^2(\mathbb{R})$.*

Proof. Any $f \in L^2(\mathbb{R})$ can be approximated by a step function, say, $g(x) = \sum_{k=1}^{N} c_k \chi_{[a_k,b_k]}(x)$ so that $\|f-g\|_{L^2} < \epsilon/2$ for any $\epsilon > 0$ (this comes from the real analysis, for example see Rudin's or Royden's "Real analysis"). Next, smooth out the corners of g so that $h \in \mathbb{C}_c^2(\mathbb{R})$ would be close to g: $\|g-h\|_{L^2} < \epsilon/2$. (One can do that for example by considering a step function of the unit interval $g(x) = \chi_{[0,1]}(x)$ and a new "smoothed out" function $h_\delta(x) = 1 + x/\delta - \frac{1}{2\pi}\sin(2\pi(x+\delta)/\delta)$ for $-\delta < x < 0$, 1 for $0 \leq x \leq 1$, $1-(1-x)/\delta - \frac{1}{2\pi}\sin(2\pi(1+\delta-x)/\delta)$ for $0 < x < \delta$ and zero otherwise. Show that it is C_c^2 and close to g if $\delta > 0$ is small.) □

Proposition 2.8. *Let $f \in L^2(\mathbb{R})$. Suppose $f_n, \hat{f}_n \in L^1(\mathbb{R})$ for each $n \in \mathbb{N}$ and $f_n \to f$ in the $L^2(\mathbb{R})$ norm. Then $\lim_{n\to\infty} \hat{f}_n$ exists a.e. and unique, i.e. does not depend on a sequence $\{f_n\}$ converging to f.*

Proof. (By the previous proposition we can choose f_n's to be C_c^2.) We have $\|\hat{f}_n - \hat{f}\|_{L^2} = \sqrt{2\pi}\|f_n - f\|_{L^2} \longrightarrow 0$ and so $\{\hat{f}_n\}$ is Cauchy in L^2. Hence, (L^2 is complete) there exists $F \in L^2$ such that $\hat{f}_n \to F$. Suppose some other sequence $\{g_n\} \to f$ and $g_n \neq f_n$. Denote by G the limit of $\hat{g}_n$'s. Then $\|F-G\| = \|F-\hat{f}_n\| + \|\hat{f}_n - \hat{g}_n\| + \|\hat{g}_n - G\|$, where each term goes to 0 (the second one is equal to $\sqrt{2\pi}\|f_n - g_n\| \to 0$ as $n \to \infty$). Thus, $F = G$ a.e. □

10. FOURIER TRANSFORM ON $L^2(\mathbb{R})$.

Definition 2.9. *Let $f \in L^2(\mathbb{R})$. Take a sequence $\{f_n\} \in L^1(\mathbb{R})$ such that $f_n \to f$ in the L^2 norm and $\hat{f}_n \in L^1(\mathbb{R})$. (One can construct such f_n's using the above two Lemmas.) Define the Fourier transform of f by*

$$\hat{f} = \lim_{n\to\infty} \hat{f}_n,$$

where limit is understood in the L^2 sense.

Similarly, define the inverse Fourier Transform $f^\vee = \lim_{n\to\infty} f_n^\vee$.

Remark. All the properties of the Fourier transform on L^1 can be extended to the L^2 functions (such as Parseval and Plancherel identities) by using the convergence argument. Moreover, note that $\hat{}\ :\ L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is one-to-one and onto, with the inverse defined above.