LECTURE NOTES FOR MATH 494, SPRING 2005

1. Preliminaries from Analysis

1. Inner Product.

Definition 1.1. *Let V be a vector space over $\mathbb{C}$. Then*

$$\langle \cdot, \cdot \rangle : \; V \times V \; \longrightarrow \mathbb{C}$$

is an inner product in V if

(i) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle \;\; \forall \; u, v, w \in V$ *(Additivity)*
(ii) $\langle \alpha u, v\rangle = \alpha \langle u, v\rangle \;\; \forall \; u, v \in V, \alpha \in \mathbb{C}$ *(Scalar multiplication)*
(iii) $\langle u, v\rangle = \overline{\langle v, u\rangle} \; \forall \; u, v \in V$ *(Conjugate symmetry)*
(iv) $\langle u, u\rangle \geq 0 \; \forall \; u \in V$ *(Positive definiteness)*
and $\langle u, u\rangle = 0 \iff u = 0$

Remark 1.2. *Note that from (ii) and (iii) it follows that* $\langle u, \alpha v\rangle = \overline{\alpha}\langle v, u\rangle \; \forall u, v \in V, \alpha \in \mathbb{C}$.

Example. The dot product $z \cdot w = \sum_{i=1}^{n} z_i \overline{w_i}$ is an inner product in $\mathbb{C}^n$.

2. Examples of basic function spaces and their properties.

Definition 1.3. *Let $\{z_j\}_{j=1}^{\infty}$ be a sequence in $\mathbb{C}$. Define*

$$l^p(\mathbb{N}) = \left\{ : \; \sum_{j=1}^{\infty} |z_j|^p < \infty \right\}.$$

In particular, $l^2(\mathbb{N})$ consists of square-summable sequences (i.e., $\sum_{j=1}^{\infty} |z_j|^2 < \infty$).

Inner product on $l^2(\mathbb{N})$:

$$\langle z, w\rangle = \sum_{j=1}^{\infty} z_j \overline{w_j}.$$

Definition 1.4. *Let*

$C([0,1]) = \{$*set of continuous (complex-valued) functions on*$[0,1]\}$.

Inner product on $C([0,1])$:

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\, dx.$$

3. Norm

Definition 1.5. *Define the norm on an inner product vector space V as*

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

Lemma 1.6. (Cauchy-Schwarz Inequality)
For any $u, v \in V$ we have

(1) $$| \langle u, v\rangle | \leq \|u\| \, \|v\|.$$

Proof. Assume $v \neq 0$ (otherwise (1) holds automatically). Let $\lambda \in \mathbb{C}$. Then

$$\langle u + \lambda v, u + \lambda v\rangle \geq 0,$$

and so,

$$\langle u, u\rangle + \lambda \langle v, u\rangle + \overline{\lambda} \langle u, v\rangle + \lambda \cdot \overline{\lambda} \langle v, v\rangle = \|u\|^2 + \lambda\overline{\langle u, v\rangle} + \overline{\lambda} \langle u, v\rangle + |\lambda|^2 \|v\|^2.$$

Choose $\lambda = -\dfrac{\langle u, v\rangle}{\|v\|^2}$ and substitute in the above:

$$\|u\|^2 - 2\frac{\langle u, v\rangle\,\overline{\langle u, v\rangle}}{\|v\|^2} + \frac{\langle u, v\rangle\,\overline{\langle u, v\rangle}}{\|v\|^4}\|v\|^2 \geq 0.$$

Simplifying, we get

$$\|u\|^2\|v\|^2 \geq |\langle u, v\rangle|^2.$$

□

Corollary 1.7. (TRIANGLE INEQUALITY)

$$\|u + v\| \leq \|u\| + \|v\|$$

(expand $\|u + v\|^2$ and use Cauchy-Schwarz).

4. SPACE OF SQUARE-INTEGRABLE OR FINITE ENERGY FUNCTIONS.

Definition 1.8. *Let $\Omega \subseteq \mathbb{R}$. Then*

$$L^2(\Omega) = \left\{ f : \Omega \to \mathbb{C} \text{ such that } \int_\Omega |f(x)|^2\,dx < \infty \right\}.$$

Inner product: $\langle f, g\rangle = \int_\Omega f(x)\overline{g(x)}\,dx$.
Norm: $\|f\|_{L^2(\Omega)} = \left(\int_\Omega |f(x)|^2\,dx\right)^{1/2}$.

Note that an inner product is well-defined, since by Cauchy-Schwarz

$$|\langle f, g\rangle| \leq \|f\|_{L^2(\Omega)}\,\|g\|_{L^2(\Omega)} < \infty.$$

The most used spaces are $L^2(\mathbb{R})$, $L^2([0, 1])$ and $L^2([0, 2\pi])$.

5. SPACE OF ABSOLUTELY INTEGRABLE FUNCTIONS.

Definition 1.9. *Let $\Omega \subseteq \mathbb{R}$. Then*

$$L^1(\Omega) = \left\{ f : \Omega \to \mathbb{C} \text{ such that } \int_\Omega |f(x)|\,dx < \infty \right\}.$$

Remark 1.10. *L^1 does not have an inner product associated with the space. So it is not an inner product vector space. However, one can define a "norm" on L^1, i.e. a functional which will have the same properties as the norm generated by the inner product, i.e.*

- $\|v\| \leq 0$ *and* $\|v\| = 0 \iff v = 0$
- $\|\alpha v\| = |\alpha|\|v\|, \alpha \in \mathbb{C}$
- $\|v + u\| \leq \|v\| + \|u\|$ *(triangle inequality)*

Taking into account the above remark, define

$$\|f\|_{L^1(\Omega)} = \int_\Omega |f(x)|\,dx.$$

If Ω has a finite Lebesgue measure ($m(\Omega) < \infty$), then any $f \in L^2(\Omega)$ also belongs to $L^1(\Omega)$, since

$$\|f\|_{L^1(\Omega)} = \int |f(x)|\,dx = \langle f, sgnf\rangle \leq \|f\|_{L^2(\Omega)}\|sgnf\|_{L^2(\Omega)} = \|f\|_{L^2(\Omega)}[m(\Omega)]^{1/2},$$

by Cauchy-Schwarz.

The opposite is not necessarily true. For example, take

$$f(x) = \begin{cases} 1/x & x \geq 1 \\ 0 & x < 1 \end{cases}.$$

Then $\|f\|_{L^2(\mathbb{R})} = 1$ but $\|f\|_{L^1(\mathbb{R})} = \infty$, so $f \in L^2(\mathbb{R})$ but $f \notin L^1(\mathbb{R})$.

6. Spaces L^p of integrable functions.

We can generalize the previous two spaces to any p.

Definition 1.11. *Let $\Omega \subseteq \mathbb{R}$ and $1 \leq p < \infty$. Then*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{C} \text{ such that } \int_\Omega |f(x)|^p \, dx < \infty \right\}.$$

If $p = \infty$, we define L^∞ as a space of (essentially) bounded functions, i.e. functions bounded almost everywhere on Ω:

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{C} \text{ such that } \sup\{M : m\{x \in \Omega : |f(x)| > M\} > 0\}\} .$$

These are also not an inner product spaces (except for $p = 2$), but the norm can be defined as

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p} .$$

The important inequalities for these spaces are

- Minkowski's inequality
  For any $f, g \in L^p$,
  $$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$
- Minkowski's integral inequality
  For any $f \in L^p$,
  $$\left\| \int f(x, \cdot) \, dx \right\|_{L^p} \leq \int \|f(x, \cdot)\|_{L^p} \, dx.$$
- Hölder's inequality
  For any $f \in L^p$ and $f \in L^q$ (with $\frac{1}{p} + \frac{1}{q} = 1$),
  $$\left| \int f(x) g(x) \, dx \right| \leq \|f\|_{L^p} \cdot \|g\|_{L^q}.$$
  (Here, $p = 1$ is allowed with $q = \infty$.)

7. Orthonormal bases in the Hilbert spaces.

Definition 1.12. *The Hilbert space $\mathcal{H}$ is a complete inner product space (i.e. every Cauchy sequence converges to an element of $\mathcal{H}$ in a norm generated by the inner product).*

Spaces L^2, l^2, $\mathbb{R}$, $\mathbb{C}$ are Hilbert spaces; L^p, $p \neq 2$ is not.

Definition 1.13.

- *Two functions f and g are orthogonal ($f \perp g$) if $\langle f, g \rangle = 0$.*
- *A sequence $\{v_n\}$ is an orthogonal sequence if $\langle v_n, v_m \rangle = 0$ for $n \neq m$.*
- *A sequence $\{v_n\}$ is an orthonormal sequence if*
  $$\langle v_n, v_m \rangle = \delta_{nm} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} .$$
- *The sequence $\{v_n\}$ is an orthonormal basis in $\mathcal{H}$ if $\{v_n\}$ is a complete o.n. sequence, i.e. an o.n. sequence with the property that the only element $g \in \mathcal{H}$ such that $\langle g, v_n \rangle = 0$ for all $n \in \mathbb{Z}$ is $g = 0$.*

- *Let* $\{v_n\}$ *be a sequence in* $\mathcal{H}$*. Then* $\sum_n v_n$ *converges in* $\mathcal{H}$ *if the symmetric partial sums*
$$s_N = \sum_{n=-N}^{N} v_n$$
converge to some element $s \in \mathcal{H}$*.*

A note about completeness: recall that in the previous lecture we showed that $\{h_{jk}\}_{j,k\in\mathbb{Z}}$ form an o.n. basis in $L^2(\mathbb{R})$ and completeness of the Haar system followed from that fact that $\overline{\text{span}\{h_{jk}\}_{j,k\in\mathbb{Z}}} = L^2(\mathbb{R})$. How does this relate with the completeness definition given above? Suppose there exists a function $f \in L^2(\mathbb{R})$ such that $\langle f, h_{jk}\rangle = 0$ for all $j, k \in \mathbb{Z}$. Then $f \perp \text{span}\{h_{jk}\}_{j,k\in\mathbb{Z}}$, i.e. $f \notin \overline{\text{span}\{h_{jk}\}} = L^2(\mathbb{R})$ contrary to $f \in L^2(\mathbb{R})$. This resolves if $f \equiv 0$.

Now we'll investigate properties of the orthonormal sequences which are not neccessarily complete.

Proposition 1.14. *If* $\{v_n\}_{n\in\mathbb{Z}}$ *is an o.n. sequence in* $\mathcal{H}$ *and* $\{a_n\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$*, then*

- $\sum_{n\in\mathbb{Z}} a_n v_n$ *converges in* $\mathcal{H}$

and

- $\left\| \sum_{n\in\mathbb{Z}} a_n v_n \right\|_{\mathcal{H}}^2 = \sum_{n\in\mathbb{Z}} |a_n|^2.$

Proof. For the first part consider partial sums $s_N = \sum_{|n|\le N} a_n v_n$. For $N > M$ we have
$$\|s_N - s_M\|^2 = \left\| \sum_{M<|n|\le N} a_n v_n \right\|^2 = \sum_{M<|n|\le N} |a_n|^2$$
(by orthogonality and finiteness of sum). Since $\{a_n\} \in l^2(\mathbb{Z})$, $\sum_n |a_n|^2 < \infty$ and the tail of this series is vanishing to zero. Thus, for any given we can choose N, M such that $\sum_{M<|n|\le N} |a_n|^2 < \epsilon$ and so $\{s_N\}$ is Cauchy. Since $\mathcal{H}$ is complete, there exists $s \in \mathcal{H}$ such that $s_N \to s$ (in $\mathcal{H}$ norm), and thus, $\sum_{n\in\mathbb{Z}} a_n v_n \to s$.

For the second part, write
$$\|s\| \le \|s_N\| + \|s - s_N\| \le \left(\sum_{|n|<N} |a_n|^2 \right)^{1/2} + \|s - s_N\|.$$
Taking a limit and using $s_N \to s$ (in $\mathcal{H}$ norm), we obtain
$$\|s\| \le \lim_{N\to\infty} \left(\sum_{|n|\le N} |a_n|^2 \right)^{1/2} = \left(\sum_{n\in\mathbb{Z}} |a_n|^2 \right)^{1/2}.$$
For the opposite inequality, use $\|s_N\| \le \|s\| + \|s_N - s\|$. □

Lemma 1.15. (BESSEL'S INEQUALITY)
Let $f \in \mathcal{H}$ and $\{v_n\}$ be an o.n. sequence in $\mathcal{H}$. Then

$$\sum_n |\langle f, v_n\rangle|^2 \le \|f\|^2_{\mathcal{H}}.$$

Proof. Consider partial sums $s_N = \sum_{|n|<N} \langle f, v_n\rangle v_n$. We have

$$\|f - s_N\|^2 = \langle f - s_N, f - s_N\rangle = \|f\|^2 - \langle f, s_N\rangle - \langle s_N, f\rangle + \|s_N\|^2.$$

Since $\langle f, s_N\rangle = \left\langle f, \sum_{|n|<N} \langle f, v_n\rangle v_n \right\rangle = \sum_{|n|<N} |\langle f, v_n\rangle|^2$, we get

$$0 \le \|f - s_N\|^2 = \|f\|^2 - \sum_{|n|<N} |\langle f, v_n\rangle|^2,$$

and taking a limit as $N \to \infty$, we obtain Bessel's inequality. □

8. MAIN PROPERTIES OF ORTHONORMAL BASES.

The completeness of an orthonormal sequence (i.e., having an o.n. basis) extends the above properties in the following way.

Theorem 1.16. *Let $\{v_n\}$ be an o.n. sequence in $\mathcal{H}$. Then $\{v_n\}$ is an o.n. basis of $\mathcal{H}$ if and only if*

$$\forall\ f \in \mathcal{H} \qquad f = \sum_{n\in\mathbb{Z}} \langle f, v_n\rangle v_n \tag{2}$$

(where convergence is in $\mathcal{H}$ sense).

Proof. Given f, denote $g = \sum_n \langle f, v_n\rangle v_n$ (this series converges by the proposition and lemma above). Then

$$\langle g, v_m\rangle = \sum_n \langle f, v_n\rangle \langle v_n, v_m\rangle = \langle f, v_m\rangle,$$

and so $\langle g - f, v_n\rangle = 0$ for any $n \in \mathbb{Z}$, which means $g = f$ (by completeness).
For the other direction we only need to show that $\{v_n\}$ is complete, which trivially follows from (2). □

Using the above decomposition, we can formulate the completeness of an o.n. sequence in 3 equivalent different ways:

Theorem 1.17. *Let $\{v_n\}$ be an o.n. sequence in $\mathcal{H}$.*

Then the following statements are equivalent:

(a) $\{v_n\}$ *is an o.n. basis in $\mathcal{H}$ (completeness)*

(b) $\langle f, g\rangle = \sum_n \langle f, v_n\rangle \overline{\langle g, v_n\rangle} \quad \forall f, g \in \mathcal{H}$ *(Parseval's relation)*

(c) $\|f\|^2_{\mathcal{H}} = \sum_n |\langle f, v_n\rangle|^2 \quad \forall f \in \mathcal{H}$ *(Plancherel's identity)*

Note that the part (c) of the Theorem is an improvement of Bessel's inequality. Here, completeness gives the exact equality.