

The sum of the inverted squared positive integers is pi squared, divided by six:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \dots = \frac{\pi^2}{6}$$

This is a remarkable result, and was discovered by Leonhard Euler, among his thousands of other interesting discoveries. The demonstration of this sum requires just a knowledge of the Taylor Series for the sine function, and a lot of cleverness.

The Taylor Series for the sine function is as follows:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

Divide this by x :

$$\frac{\sin x}{x} = 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + \dots$$

Note that each term in the Taylor series has been reduced by a power of one. Furthermore, note the following two features of the function $\frac{\sin x}{x}$:

- $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. This is easily shown via l'Hopital's Rule.
- The function $\frac{\sin x}{x}$ has roots wherever the numerator equals 0, which is $\pm\pi, \pm2\pi, \pm3\pi, \dots$, except when $x = 0$, in which the function does not exist, but has a limit of 1 (there is a deleted point (0,1) in the graph).

The function $\frac{\sin x}{x}$ can be "factored" by creating factors of the form $\left(1 - \frac{x}{a}\right)$, where a is a root. For example, since π is a root, then $\left(1 - \frac{x}{\pi}\right)$ is a factor: if you let $x = \pi$, then the factor becomes $\left(1 - \frac{\pi}{\pi}\right) = (1 - 1) = 0$. Here's a key aspect of this demonstration: since the roots comes in positive and negative pairs, we can create the following factors along with the logical simplification:

- Roots $\pm\pi$ create factors $\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)$, which multiplies to $\left(1 - \frac{x^2}{\pi^2}\right)$.
- Roots $\pm2\pi$ create factors $\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)$, which multiplies to $\left(1 - \frac{x^2}{4\pi^2}\right)$.
- Roots $\pm3\pi$ create factors $\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)$, which multiplies to $\left(1 - \frac{x^2}{9\pi^2}\right)$.
- And so forth.

Therefore, the function $\frac{\sin x}{x}$ can be written as an infinite product of factors:

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(1 - \frac{x^2}{16\pi^2}\right)\dots \quad (x \neq 0)$$

This factored form obviously accounts for all roots, and cleverly accounts for the case when $x = 0$, in which case the right side becomes an infinite product of 1's, which is just 1, which agrees with the limit value of the function when x approaches 0.

Now the factors must be fully expanded by multiplication. Instead of taking all possible products, we only want the ones that leave an x^2 term. Note that this can only be done by taking one such term (for example, taking the first one, $-\frac{x^2}{\pi^2}$), and multiplying it by all the 1's. Then doing it again for the next x^2 term, and multiplying that by all the 1's, and so forth... Thus, the coefficient of the x^2 term of this huge multiplication will be

$$\left(-\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2} - \frac{1}{16\pi^2} - \dots\right)x^2$$

This term is compared to the term found in the Taylor Series for $\frac{\sin x}{x}$, which is $-\frac{1}{3!}x^2$:

$$\left(-\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2} - \frac{1}{16\pi^2} - \dots\right)x^2 = -\frac{1}{3!}x^2$$

This implies

$$-\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2} - \frac{1}{16\pi^2} - \dots = -\frac{1}{3!}$$

The negative cancel and 3! is 6:

$$\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \dots = \frac{1}{6}$$

Multiply both sides by π^2 , and the result is now complete:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots = \frac{\pi^2}{6}$$

This same process can be used to find the sum of the inverted fourth powers of the positive integers. You could show by a similar construction that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \dots = \frac{\pi^4}{90}$$

In fact, all sums of inverted *even* powers of the positive integers can be calculated this way. What's interesting is that there is no known "closed form" sum for the inverted cubes of the positive integers. No one to date has shown an exact sum for

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \dots = ?$$

You could be very famous if you discover such a sum!