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## **Special functions associated with the quantum dynamical Yang-Baxter equation**

*Abstract*

Let  $g$  be a complex simple Lie algebra (for instance  $sl(2, C)$ ). Let  $h$  be a Cartan subalgebra in  $g$ , let  $h^*$  be its linear dual and make a choice of positive roots. Let  $U_q g$  be the corresponding quantized universal enveloping algebra. Let  $M_\lambda$  be the Verma module for  $g$  or for  $U_q g$  of highest weight  $\lambda$  in  $h^*$ .

With a tensor product of a Verma module and a finite dimensional module one can associate an *intertwining operator* which generalizes the notion of Clebsch-Gordan coefficients. From that one can build a *fusion matrix* and an *exchange matrix* associated with the tensor product of a Verma module and two finite dimensional modules. The exchange matrix generalizes the notion of Racah coefficients. The intertwining operator, fusion matrix and exchange matrix depend rationally on  $\lambda$  in  $h^*$  (or on  $q^\lambda$  in the quantum case). The fusion matrix  $J_{VW}(\lambda)$  and the exchange matrix  $R_{VW}(\lambda)$  are  $h$ -intertwining operators on the tensor product of the finite dimensional  $g$ -modules  $V$  and  $W$ , and they satisfy some important algebraic identities which also involve shifts in  $\lambda$ . This culminates into the *quantum dynamical Yang-Baxter equation* (QDYBE), which generalizes the well known quantum Yang-Baxter equation.

In certain cases, certainly for  $sl(2)$ , the intertwining operator, fusion matrix and exchange matrix can be computed explicitly as special functions, and the general formulas produced by the above theory yield formulas for these special functions.

A further construction, starting with the intertwining operator, yields the *weighted trace functions* depending on  $\lambda$  and  $\mu$  in  $h$ , while a construction starting from the exchange matrix yields a commuting family of difference operators. For  $sl(n)$  this yields in the quantum case Macdonald polynomials for root system of type  $A$ , and also the Macdonald-Ruijsenaars and dual Macdonald-Ruijsenaars eigenvalue equations for the Macdonald polynomials, as well as a symmetry in  $\lambda$  and  $\mu$ . The weighted trace functions also satisfy the so-called qKZB and dual qKZB equation, while there are limit cases to the qKZ and KZ equations.

The material sketched above found its origin in physics and was developed during the last 15 years by various physicists and mathematicians. A nice survey can be found in P. Etingof and O. Schiffmann, math.QA/9908064, while more details and proofs concerning the various difference equations satisfied by the weighted trace functions are given in Etingof and Varchenko, math.QA/9907181.

The lecture will survey these results with emphasis on special function aspects. Only a very minor part of this lecture will contain original results by the speaker (and his co-worker N. Touhami).

In particular during the last few years there has been a fast development, which particularly aims at the further generalization to elliptic quantum groups and dynamical quantum groups, and to a generalization of Macdonald polynomials in the elliptic case. These newer developments will be discussed in the second lecture by Koelink at this Advanced Study Institute. The present lecture will be a useful preparation for Koelink's lecture.