

HARMONIC ANALYSIS ON QUANTUM GROUPS AND SPECIAL FUNCTIONS

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1. INTRODUCTION

By now it is well established that there is an intimate relationship between representation theory of groups on the one hand and special functions on the other hand. This relationship extends in numerous ways, to various types of groups and various types of special functions.

In the last 15 years it has become clear that there exists an intimate relationship between quantum groups and special functions of basic hypergeometric type benefitting both sides. The theory and results are particularly satisfactory in the compact quantum group cases. In these cases the special functions are usually polynomials and most constructions are algebraic. A tremendous amount of information is then available.

In these lectures we have a look at two possible future directions. In the first lecture we discuss a scheme of transforms associated to the simplest non-compact semisimple quantum group, the quantum analogue of $SU(1, 1)$. This scheme is a non-polynomial extension of the q -analogue of Askey's scheme of orthogonal polynomials of basic hypergeometric type. We focus on the special function side of this story, and the explicit scheme together with its limit transitions will be discussed. This part is joint work with Jasper Stokman.

In the second lecture we want to draw your attention to an extension of the notion of quantum group to that of a dynamical quantum group. The adjective 'dynamical' comes from the fact that dynamical quantum groups are related to solutions of the dynamical Yang-Baxter equation, whereas quantum groups are related to constant solutions of the Yang-Baxter equation. We study the algebras associated with one of the simplest solutions, namely the trigonometric solution for the case $\mathfrak{sl}(2)$. Then we can already associate special functions, such as the Askey-Wilson polynomials, to these algebras. Motivated by this simplest example, or toy model, we expect to obtain new insights for special functions from the dynamical quantum groups. This part is joint work with Hjalmar Rosengren.

This abstract is not meant to give a full introduction to these subjects. More information can be found in the papers mentioned and their references. Other talks at this conference on closely related subjects are the lectures by Koornwinder, Rahman, Rosengren, Ruijsenaars, Stokman, Suslov, Tolstoy, Zhedanov.

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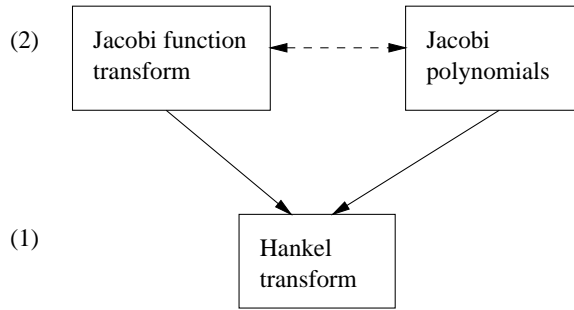


FIGURE 1. Jacobi function scheme.

2. THE ASKEY-WILSON TRANSFORM SCHEME

This section gives a short introduction to and overview of the first talk, which is based on joint work with Jasper Stokman [7], [8], [9], [10]. Further information and references to related papers by a.o. Ismail, Kakehi, Koornwinder, Masuda, Masson, Suslov, Swarttouw can be found in these papers.

The relationships between the Jacobi polynomials, the Jacobi function transform and the Hankel transform is depicted in Figure 1. Here the Jacobi polynomials are the orthogonal polynomials on the interval $[-1, 1]$ with respect to the beta measure. The Jacobi functions are non-terminating hypergeometric functions and the Jacobi function transform is an integral transform on a half-line with the Jacobi function as the kernel. The inverse transform is known explicitly. Similarly, the Hankel transform is an integral transform with the Bessel function as its kernel, and its inverse is of the same form.

In Figure 1 the Jacobi functions and Jacobi polynomials have 2 degrees of freedom and the Bessel functions have 1 degree of freedom. The Jacobi function, considered as a function of its spectral parameter, is an analytic continuation of the Jacobi polynomial in its degree. The Bessel function can be obtained by a limiting procedure from the Jacobi function as well as by a limiting procedure from the Jacobi polynomial. The corresponding transforms and orthogonality relations can be obtained from a spectral analysis of the corresponding second order differential operator.

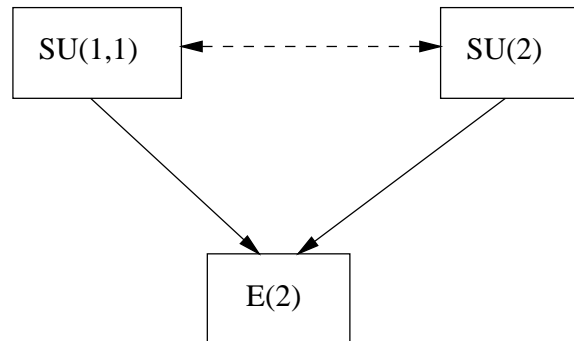


FIGURE 2. Group theoretic interpretation of the Jacobi function scheme.

The Jacobi polynomials are part of Askey's scheme of hypergeometric orthogonal

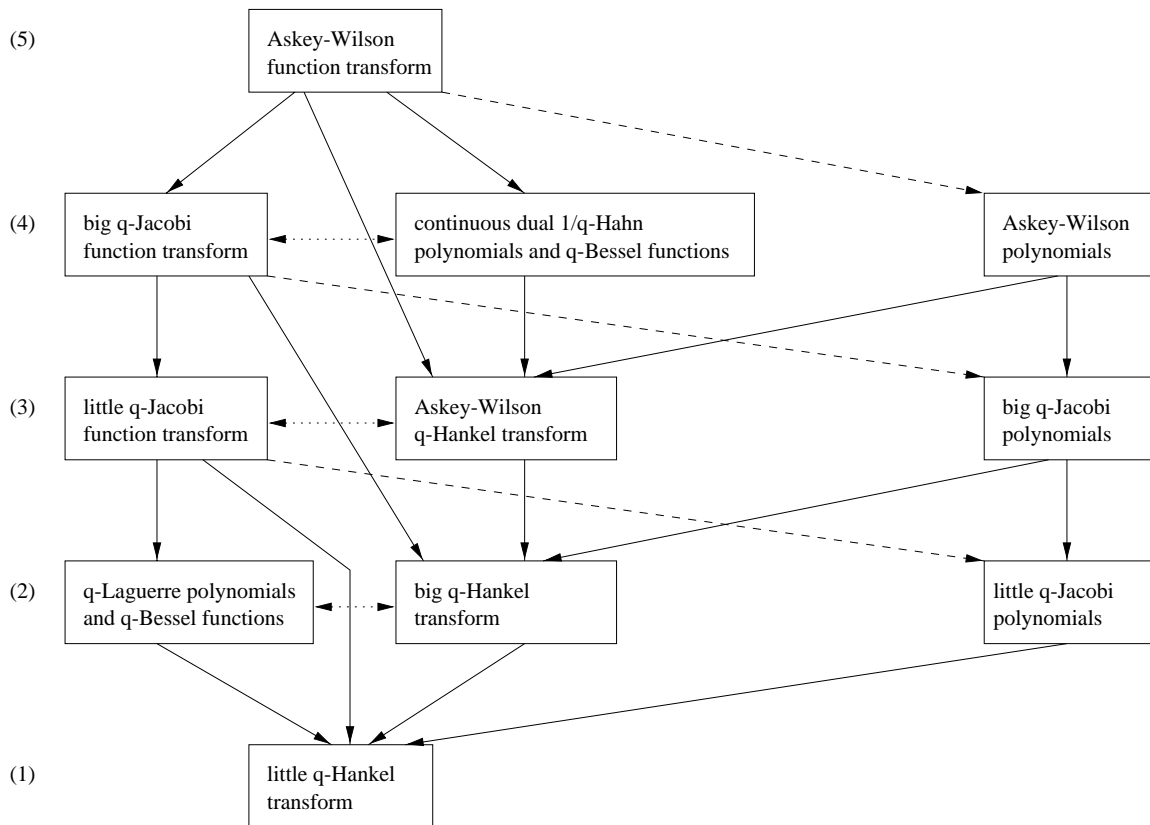


FIGURE 3. Askey-Wilson function scheme.

polynomials, see e.g. [5] for a good description of the various families in Askey’s scheme. So we see that Figure 1 gives an extension of Askey’s scheme to non-polynomial special functions of hypergeometric type. The corresponding group theoretic setting is given in Figure 2. In particular, Jacobi functions, respectively Jacobi polynomials and Bessel functions, occur as matrix coefficients of irreducible unitary representations of the Lie groups $SU(1, 1)$, respectively $SU(2)$ and $E(2)$. From this group theoretic interpretation many results can be obtained.

In Figure 3 the right three boxes containing the polynomials are part of the q -analogue of Askey’s scheme of basic hypergeometric polynomials. Figure 3 contains three analogues of Figure 1; one analogue consisting of the three boxes with the adjective ‘Askey-Wilson’, one analogue consisting of the three boxes with the adjective ‘big q -’ and one analogue consisting of the three boxes with the adjective ‘little q -’. The relations within each analogue are similar to the relations in Figure 1. However, an important difference between Figure 3 and Figure 1 is that there is an extra degree of freedom in the Askey-Wilson function transform, respectively big and little q -Jacobi function transform, compared to the Askey-Wilson polynomials, respectively big and little q -Jacobi polynomials. This is due to an extra degree of freedom in the measure of the function transforms.

There are limit transitions from the Askey-Wilson analogue of Figure 1 inside Figure 3 to the big q -analogue of Figure 1 in Figure 3 as well as limit transitions from the

big q -analogue of Figure 1 to the little q -analogue of Figure 1 in Figure 3. These limit transitions are similar to the existing limit transitions within the q -analogue of Askey's scheme of basic hypergeometric orthogonal polynomials. Moreover, the Askey-Wilson type q -Hankel transform is dual to the little q -Jacobi function transform, as indicated by the dotted horizontal line. This explains all but two of the boxes in Figure 3. The remaining two boxes are obtained by a duality argument from the boxes to which they are linked by a horizontal arrow. These boxes are related to indeterminate moment problems, since they give an orthogonal basis complementing the orthogonal basis of polynomials for an explicit non-extremal solution to the moment problem. The indeterminate moment problems correspond to the one for the q -Laguerre polynomials and the one for the continuous dual q^{-1} -Hahn polynomials. In particular, the Askey-Wilson function transform and the little q -Hankel transform are considered as self-dual transforms. For the little q -Hankel transform the inverse is indeed given by the little q -Hankel transform with the spectral and geometric parameter interchanged and for the Askey-Wilson function transform this holds after going over to a dual parameter set.

The form of the extension of the q -analogue of Askey's scheme in Figure 3 is strongly motivated by the quantum group theoretic interpretation of the special functions involved on the quantum group analogue of Figure 2. In particular, the Askey-Wilson, big and little q -Jacobi polynomials are interpreted on the quantum $SU(2)$ group, the Askey-Wilson, big and little q -Jacobi functions are interpreted on the quantum $SU(1, 1)$ group and the Askey-Wilson, big and little q -Bessel functions are interpreted on the quantum $E(2)$ group as matrix elements of irreducible unitary (co)representations. The corresponding transform pairs and orthogonality relations are related to the (spherical) Fourier transform and Schur's orthogonality relations on these quantum groups.

Although the Askey-Wilson transform scheme is motivated from its interpretation on quantum groups, this scheme can be considered from a purely analytic point of view. In particular, the various transforms and orthogonality relations can all be considered as the spectral measures for a second order q -difference operator on a suitable weighted L^2 -space. The functions involved are all eigenfunctions to such a second order q -difference operator. So the whole scheme in Figure 3 can be obtained from a spectral analysis of a (usually unbounded) symmetric operator. Moreover, all special functions involved are also eigenfunctions of a second order q -difference operator in the spectral variable.

The purpose of this talk is to discuss the Askey-Wilson function scheme of Figure 3 in detail. In particular some of the transforms will be discussed explicitly, and we discuss some of the limit transitions. There are several research questions associated to this scheme and we discuss some possibilities in that direction. See also Stokman's talk at this conference for the Hecke algebraic structure behind the Askey-Wilson transform.

3. SPECIAL FUNCTIONS AND DYNAMICAL QUANTUM GROUPS

This section gives a short introduction to and overview of the second talk, which is based on joint, yet unfinished, work with Hjalmar Rosengren, see [6]. Further introductory information on dynamical quantum groups can be found in [2], and [3].

As pointed out in the introduction, the interplay between compact quantum groups and explicit sets of orthogonal polynomials has been very fruitful. This usually involves

both the notion of quantum group as the non-commutative analogue of the algebra of functions on a (semisimple compact) Lie group and of the quantised universal enveloping algebra of a semisimple Lie algebra together with their duality. To the quantised function algebra we can already associate special functions, and in this talk we consider what can be done for one of the simplest cases of a dynamical quantum group.

First we recall that the algebraic structure of the standard deformation of the algebra of functions on the space of 2×2 -matrices can be written in terms of $RTT = TTR$ relations for a 4×4 -matrix R , see [1, Ch. 7]. It is possible to give the deformation a bialgebra structure, i.e. having a comultiplication and counit satisfying the usual properties. For the quantum $SL(2)$ group we take R , viewed as an element of $\text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, to be a specific (constant) solution of the Yang-Baxter equation $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$ as an identity in $\text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$ using the usual leg numbering notation. Then the quantum $SL(2)$ group is obtained by adding one extra relation corresponding to the requirement that the determinant equals 1. This then even becomes a Hopf algebra.

In statistical mechanics it is well-known that there exist a wealth of solutions of the Yang-Baxter equation that involve external parameters. Here we are interested in dynamical parameters, corresponding to face models in statistical mechanics. To explain the dynamical Yang-Baxter equation, let \mathfrak{h} be a finite-dimensional complex vector space, viewed as a commutative Lie algebra, and $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$ a diagonalizable \mathfrak{h} -module, so $V_\alpha = \{v \in V \mid H \cdot v = \alpha(H)v \forall H \in \mathfrak{h}\}$. In the context of dynamical quantum groups, \mathfrak{h} will typically be a Cartan subalgebra of the corresponding simple Lie algebra \mathfrak{g} , so that in our case we can think of $\mathfrak{h} = \mathbb{C}$. The dynamical Yang-Baxter equation is

$$R^{12}(\lambda - h^{(3)})R^{13}(\lambda)R^{23}(\lambda - h^{(1)}) = R^{23}(\lambda)R^{13}(\lambda - h^{(2)})R^{12}(\lambda).$$

This is an identity in the algebra of meromorphic functions $\mathfrak{h}^* \rightarrow \text{End}(V \otimes V \otimes V)$. Here $R: \mathfrak{h}^* \rightarrow \text{End}(V \otimes V)$ is a meromorphic function, h indicates the action of \mathfrak{h} , and we use the leg numbering notation. For instance, $R^{12}(\lambda - h^{(3)})$ denotes the operator

$$R^{12}(\lambda - h^{(3)})(u \otimes v \otimes w) = (R(\lambda - \mu)(u \otimes v)) \otimes w, \quad w \in V_\mu.$$

A dynamical R -matrix is by definition a solution of the dynamical Yang-Baxter equation which is \mathfrak{h} -invariant, that is, $R: \mathfrak{h}^* \rightarrow \text{End}_{\mathfrak{h}}(V \otimes V)$ or R commutes with the tensor action of \mathfrak{h} on $V \otimes V$.

It is possible to define an algebraic structure to a dynamical R -matrix following the $RTT = TTR$ -formalism resulting in so-called \mathfrak{h} -bialgebroids, and next in \mathfrak{h} -Hopf algebroids, also known as dynamical quantum groups, see [3], [2]. The ‘oid’-part comes from the fact that solutions of the classical dynamical Yang-Baxter equation are related to Poisson-Lie groupoids, see [2] and references therein.

We work with the following explicit solution to the dynamical Yang-Baxter equation. We take $\mathfrak{h} = \mathbb{C} \cdot H$ and V to be the two-dimensional \mathfrak{h} -module $V = \mathbb{C}e_+ \oplus \mathbb{C}e_-$ with $He_\pm = \pm e_\pm$. In the basis $e_+ \otimes e_+$, $e_+ \otimes e_-$, $e_- \otimes e_+$, $e_- \otimes e_-$, the dynamical R -matrix

we consider is given by

$$R(\lambda) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & \frac{q^{-1}-q}{q^{2(\lambda+1)}-1} & 0 \\ 0 & \frac{q^{-1}-q}{q^{-2(\lambda+1)}-1} & \frac{(q^{2(\lambda+1)}-q^2)(q^{2(\lambda+1)}-q^{-2})}{(q^{2(\lambda+1)}-1)^2} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

with $\lambda(H) = 1$. For more information on how to obtain this matrix as a solution to the dynamical Yang-Baxter equation from the (non-dynamical) quantum groups see [2] and [11], where different approaches are used. See Rosengren's talk at this conference for his approach [11]. If you don't want to use a quantum group, you can check by hand (or maybe better use a computer algebra package) that $R(\lambda)$ does indeed satisfy the dynamical Yang-Baxter equation. Note that formally for $\lambda \rightarrow -\infty$ the resulting matrix $R(-\infty)$ equals the R -matrix associated to the non-dynamical quantum $SL(2)$ group.

This whole algebraic machinery, which we didn't explain, leads to the following algebra; the algebra A is generated by the four generators $\alpha, \beta, \gamma, \delta$, together with two copies of the field $M_{\mathfrak{h}^*}$ of meromorphic functions on \mathfrak{h}^* , whose elements we write as $f(\lambda), f(\mu)$. The defining relations are

$$\alpha\beta = qF(\mu-1)\beta\alpha, \quad \alpha\gamma = qF(\lambda)\gamma\alpha, \quad \beta\delta = qF(\lambda)\delta\beta, \quad \gamma\delta = qF(\mu-1)\delta\gamma,$$

together with any two of the four relations

$$\begin{aligned} \alpha\delta - \delta\alpha &= H(\lambda, \mu)\gamma\beta, & G(\mu)\alpha\delta - G(\lambda)\delta\alpha &= H(\lambda, \mu)\beta\gamma, \\ \beta\gamma - G(\mu)\gamma\beta &= I(\lambda, \mu)\delta\alpha, & \beta\gamma - G(\lambda)\gamma\beta &= I(\lambda, \mu)\alpha\delta, \end{aligned}$$

and, for arbitrary $f, g \in M_{\mathfrak{h}^*}$, $f(\lambda)g(\mu) = g(\mu)f(\lambda)$, and

$$\begin{aligned} f(\lambda)\alpha &= \alpha f(\lambda+1), & f(\mu)\alpha &= \alpha f(\mu+1), \\ f(\lambda)\beta &= \beta f(\lambda+1), & f(\mu)\beta &= \beta f(\mu-1), \\ f(\lambda)\gamma &= \gamma f(\lambda-1), & f(\mu)\gamma &= \gamma f(\mu+1), \\ f(\lambda)\delta &= \delta f(\lambda-1), & f(\mu)\delta &= \delta f(\mu-1). \end{aligned}$$

Here the functions F, G, H and I are defined by

$$\begin{aligned} F(\lambda) &= \frac{q^{2(\lambda+1)} - q^{-2}}{q^{2(\lambda+1)} - 1}, \\ G(\lambda) &= \frac{(q^{2(\lambda+1)} - q^2)(q^{2(\lambda+1)} - q^{-2})}{(q^{2(\lambda+1)} - 1)^2}, \\ H(\lambda, \mu) &= \frac{(q - q^{-1})(q^{2(\lambda+\mu+2)} - 1)}{(q^{2(\lambda+1)} - 1)(q^{2(\mu+1)} - 1)}, \\ I(\lambda, \mu) &= \frac{(q - q^{-1})(q^{2(\mu+1)} - q^{2(\lambda+1)})}{(q^{2(\lambda+1)} - 1)(q^{2(\mu+1)} - 1)}. \end{aligned}$$

As an example of how the dynamical parameters play a role we state the following commutation relations valid in A ;

$$\alpha^n \gamma^m = q^{mn} \frac{(q^{-2(\lambda+m+1)}; q^2)_n}{(q^{-2(\lambda+1)}; q^2)_n} \gamma^m \alpha^n.$$

Formally replacing $\lambda, \mu \rightarrow -\infty$ gives back an algebra isomorphic to the standard quantised function algebra for $SL(2)$, apart from the determinant relation.

As it turns out, this algebra can be made into a \mathfrak{h} -Hopf algebroid by adding one extra relation, which is the analogue of the condition that the determinant should equal 1. The resulting algebra has much more structure, and in particular there exists a comultiplication that is defined on the generators by familiar formulas;

$$\begin{aligned} \Delta(\alpha) &= \alpha \otimes \alpha + \beta \otimes \gamma, & \Delta(\beta) &= \alpha \otimes \beta + \beta \otimes \delta, \\ \Delta(\gamma) &= \gamma \otimes \alpha + \delta \otimes \gamma, & \Delta(\delta) &= \gamma \otimes \beta + \delta \otimes \delta, \\ \Delta(f(\lambda)) &= f(\lambda) \otimes 1, & \Delta(g(\mu)) &= 1 \otimes g(\mu). \end{aligned}$$

The comultiplication Δ ends up in a restricted tensor product.

The existence of the comultiplication allows us to define the notion of a corepresentation for an \mathfrak{h} -Hopf algebroid. We can give a classification of irreducible finite-dimensional corepresentations and obtain an appropriate analogue of the Peter-Weyl theorem. It is possible to explicitly calculate the matrix elements of such irreducible finite-dimensional corepresentations as terminating very-well-poised ${}_8W_7$ -series and hence as Askey-Wilson polynomials. Furthermore, we can calculate the Clebsch-Gordan coefficients in terms of q -Racah polynomials, and some other type of orthogonality relations for rational functions. We can give a sufficiently large set of representations of this algebra in order to convert identities in non-commuting variables into identities involving commuting variables.

The purpose of the second talk is to discuss the notions and results above in more detail for this simplest case of a dynamical quantum group. We will emphasize the link to special functions and related results. The results for this simple model show that it is worthwhile to discuss other cases from this point of view as well. In particular for some of the elliptic solutions of the dynamical Yang-Baxter equation this might lead to an elliptic analogue of the Askey-Wilson polynomials. See also Frenkel and Turaev [4], Warnaar [12] and Zhedanov's lectures at this conference for information on elliptic hypergeometric series. See also Koornwinder's talk on special functions related to the dynamical Yang-Baxter equation.

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