FUNCTION COMPOSITION AS COMBINING TRANSFORMATIONS: LESSONS LEARNED FROM THE FIRST ITERATION OF AN INSTRUCTIONAL EXPERIMENT

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We report on an instructional experiment designed to support K-12 teachers conceiving of the composition of linear functions as combining transformations in a context involving an imaginary elevator. We discuss the design and implementation of instruction in terms of a two-phase cycle involving the formulation of an initial and then a revised local instructional theory for the concept of function composition. Our empirical results are embedded within this discussion and presented in two interrelated parts. We highlight teachers' thinking regarding addition of integers in relation to the elevator context, their thinking regarding formalizing the composition operation in relation to this context, and we articulate our principled efforts to leverage the former in support of the latter.

Introduction

The reform-based Grade 3 *Investigations Series* curricular unit *Up and Down the Number Line* (Tierney, Shulman-Weinberg, & Nemirovsky, 1995) entails a context (an imaginary elevator) designed to support children's thinking of integers as transformations and addition (implicitly) as composition. We report on our study that employed this unit as a point of departure for an instructional sequence designed to support K-12 teachers' understanding of function composition.

Our instructional sequence adapted and extended aspects of the *Investigations* unit to develop the concept of function composition in the elevator context. Our aim was to help teachers develop coherent conceptions of function composition - a concept that is not well represented in the research literature but with which college students apparently have difficulty (Engelke, Oehrtman, & Carlson, 2005).

We first describe the setting for the implementation of the instructional sequence and elaborate initial conjectures about students' conceptions and learning (Simon, 1995). We then provide selected findings from our retrospective analysis of the implementation of the instructional sequence. We discuss these findings with an eye toward revising our initial conjectures. Finally we describe the revised local instructional theory and our plan for further elaboration of it in order to support the development of the concept of function composition as combining transformations.

Methodology

Local Instructional Theory

Although we expect our project to result in the production of an instructional sequence, our overall goal is to produce something more generalizable than a specific sequence of instructional activities – a local instructional theory. The purpose of the local (content specific) instructional theory is to provide a rationale for the instructional activities that draws on the researchers' models of students' emerging and developing conceptions in relation to their engagement with

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designed instruction. Local instructional theories feature three key ingredients, the description of which is adapted from Gravemeijer (1998):

- Identification of students' informal knowledge and strategies on which the instruction can be built.
- Design principles for instructional activities that can be used to evoke these kinds of informal knowledge and strategies.
- Design principles for instructional activities that can capitalize on these informal understandings in order to meet the goals of instruction.

We use this description as a framework for our research. Our results are presented in two sections. First, we describe the teachers' (our students) informal ways of thinking about integer addition in the elevator context. We identify those ways of thinking that seemed to provide a starting point for developing the concept of function composition, and we discuss aspects of the instructional sequence that helped to evoke these ways of thinking (or seemed to evoke less productive ways of thinking). We then describe our efforts to capitalize on the teachers' ways of thinking about integer addition in the elevator context in order to develop the notion of function composition. We also describe the teachers' ways of thinking about function composition that emerged as they interacted with the instructional tasks. A retrospective analysis of this first cycle of design and implementation guided the development of a revised instructional sequence (and conjectures about the anticipated learning) on the way to developing a local instructional theory.

Setting and Participants

We engaged 4 cohorts of K-12 teachers with the sequence of instructional tasks over four 2-hour-long sessions occurring on consecutive days. The sequence was part of a course designed for a three-week summer residential institute for K-12 mathematics teachers. Instruction generally featured an inquiry-based approach to concept development. The tasks were designed to evoke the participants' informal understandings and strategies and to leverage these as the foundation for the development of the more formal or conventional mathematics. Activities typically began by having teachers first consider a problem or issue in private and then asking them to share their thinking with a partner in anticipation of small group interactions and whole class discussions. Participants were encouraged to compare and contrast ideas, to question ideas and explanations, and to offer or ask for elaborations of ideas. These classroom interactions were captured with two video cameras. The written work of individual teachers was digitally photographed, as were the posters created by each group.

Preliminary Local Instructional Theory

Step 1: Thinking of Adding Integers as Combining Changes (The Elevator Context).

The exploration of the concept of function composition was grounded in the *Investigations* curricular unit already mentioned (Tierney, Shulman-Weinberg, & Nemirovsky, 1995). This unit develops the idea of net change in a context involving an elevator in an imaginary skyscraper that extends infinitely in both vertical directions. The elevator's push buttons are labeled with integers representing *changes* (magnitude and direction) in position rather than positions (floors of the building). Thus, pressing a particular button can be seen as making the elevator move that many floors up or down the skyscraper, depending on the sign of the button's numeral. The curricular unit employs this context to emphasize thinking of integers as transformations (changes) in position and the chaining together of such transformations to obtain a net change in the elevator's position.

Our plan was to have teachers first conceptualize the operation of adding integers as the operation of combining changes. The rationale for starting the sequence in this way was two-fold. First, this way of thinking about integers involves implicitly thinking of integers as functions. We expected this way of thinking to support both the transition to formulating the changes as functions and the eventual formulation of function composition as a way to combine changes. Second, we expected that the teachers' understanding of addition as a binary operation (one that takes two integers as inputs and produces an integer as an output) would support their thinking of function composition as an operation that takes two functions as inputs and produces a function as an output.

Step 2: Thinking of Changes as Functions

The second part of our instructional plan was to have teachers formulate each change (or change button) in the elevator context as a function. This was to be done by first asking the teachers to articulate the relationship between a starting floor, a change button, and the resulting ending floor (i.e. START + CHANGE = END). The next step was to introduce function notation as a way to capture this relationship (for a specific button). For example, the change button, +2, can be associated with the function $f_{+2}(S) = S + 2$. Our intent was for the teachers to associate this notation with 1) the change (process) associated with a given change button and 2) the number (object) used to label the button. Thus, the goal was to support the teachers' ability to conceive of functions as both processes (that can be combined to form other processes) and objects (that can be combined to form other objects).

Step 3: Thinking of Composing Functions as a Way to Combine Changes

Our overall goal of instruction was to have the teachers develop the idea of function composition as the linking of processes in order to produce another process. The final step of our instructional plan was to engage the teachers in thinking about how to formulate a function associated with a combination of two change buttons. Our strategy entailed having teachers think of the combination of changes as a two-step process in which the result of the first step (the ending floor after pressing the first button) is seen as the starting point for the second step (the starting floor before pressing the second button). We conjectured that this line of reasoning would support their thinking of substituting the formal rule for the first function (say, $f_{+3}(S) = S + 3$) into the rule for the second function (say, $f_{+2}(S) = S + 2$) in order to produce the new function rule, $f_{+5}(S) = S + 5$, associated with the combination of the two buttons. This intended line of reasoning is expressed by the following string of equalities: $f_{+2}(f_{+3}(S)) = f_{+2}(S + 3) = (S + 3) + 2 = S + 5 = f_{+5}(S)$.

Going into the implementation phase, we were aware that it is possible to produce a formula for such a combination simply by combining two changes to obtain a single change and then writing a function rule for this change. This approach does not involving thinking of function composition in the way we intended because it does not employ two function rules to produce the new function. Instead, the two changes are used to find the new change, which is then used to produce the new function. (This distinction will become clearer when we discuss our results.) In order to focus the teachers' attention on the process of composing functions as we intended, we asked them to describe a mathematical way to combine the two function rules in order to obtain the new function.

In sum, our overall goal in this stage of the instructional sequence was to have the teachers develop the notion of function composition (and the associated symbolic operation of substitution) as a way to formalize the process of linking changes. The remainder of the paper will focus on the first and third steps of our instructional plan.

Results Part 1: Ways of Thinking about Adding Integers in the Elevator Context

As a first task, we asked the teachers to describe what it meant to add two integers in the elevator context. This task turned out to be quite challenging for them: They tended to impose their existing view of integers as positions on the number line onto the curricular unit. Most described an integer exclusively as representing a floor of the building and not as a change in position. This interpretation had a dramatic impact on the teachers' ability to make sense of addition in the elevator context. From our perspective, the process of successively pushing two change buttons corresponds to adding two integers in the elevator context and so addition can be seen (at least implicitly) as the composition of functions. This way of thinking provides a good point of departure for developing the notion of function composition. It is also reasonable to think of integer addition in this context as combining (adding) a change and a position to obtain a new position. This way of thinking supports conceiving of a specific button as a function, but does not provide an informal way to think about function composition in the context. However, many of the teachers struggled to develop a different interpretation - one in which *both* addends were positions. The teachers were not able to generate coherent interpretations of this type. While they interpreted integers as floor positions, they associated the operation of addition with the process of moving from one floor to another (see Alice's response below). We coded the teachers' responses to capture the apparent structure of their ways of thinking about adding integers in the elevator context. The three most common ways of thinking are presented below:

Way of Thinking	*Example Response*
Integers are Floors: Addends are both floors, addition is loosely associated with moving from one floor to another.	*Alice:* Adding two integers is like riding the elevator up from a start to an ending floor where the integers are the floor positions.
Adding a Change to a Position: The start floor and the change are the addends. The sum is the ending floor.	*Erika:* If you add the net change to the starting floor, you will get your ending floor.
Combining Changes: Two changes are combined resulting in a net change.	**Pam:** ***It is the combination of 2 movements of the elevator. It is the result after the elevator has gone through 2 movements.***

Discussion Part 1: Implications for a Revised Local Instructional Theory

Our analysis suggests that a modification of the starting point for the instructional sequence is in order. In retrospect, the focus on integers and integer addition seemed to be unhelpful because it appeared to evoke the teachers' conceptions of integers as positions *not* changes in position. However the change elevator context itself did turn out to be a productive one for developing the notion of function composition. Thus, we conjecture that it might be more productive to focus teachers' attention initially on combining changes in the elevator context and not on interpreting the meaning of integer addition in this context. One possible argument against such a revision is that the teachers' conceptions of integer addition as an operation would not be tapped to support the development of function composition as an operation. However, it seems likely that 1) the teachers can think of the combination of change buttons as an operation and that this could support the development of function composition as a mathematical operation and 2) the teachers might still spontaneously draw on their understanding of integer addition even without our explicit attention to the fact that this system is isomorphic to the integers under addition.

Results Part 2: Ways of Thinking about Composing Functions in the Elevator Context

The final task situated within the elevator context involved representing each elevator change button as a linear function and thinking about how to create such a representation for a combination of two change buttons. After the teachers had some experience representing individual change buttons as functions, we asked them to consider the two-button changes given by the functions $f_{-5}(x) = x - 5$ and $f_{+3}(x) = x + 3$ and to write a single function rule to express the combination of these two changes. Because the study was conducted in a classroom setting, for the most part it was not possible to track individual students' learning as they worked on these tasks. However, by considering snapshots of different individuals' ways of thinking we are able to identify steps along a possible learning trajectory.

Combining the Changes and then Constructing the Function

Denise created the function for the combination of changes by first combining these changes to get a net change and then writing the function associated with this change:

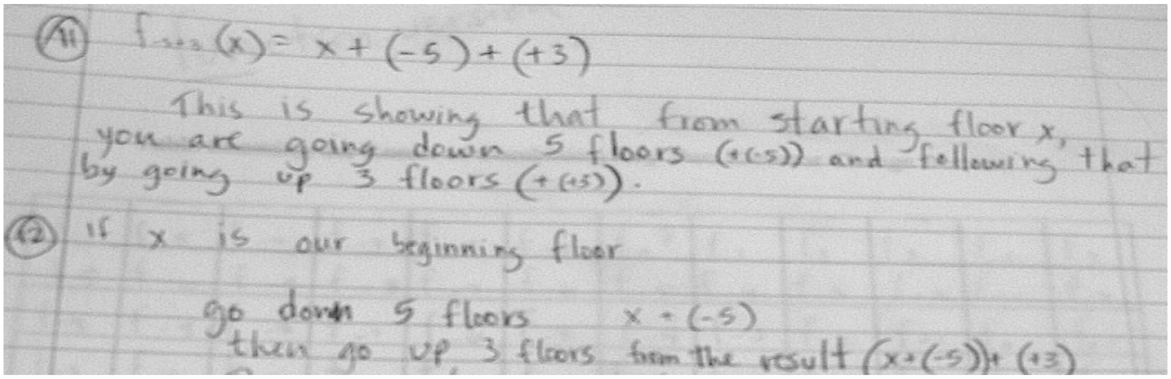
Denise: So it went down by 5 floors […] it started at any floor, and then came back up 3 […] and the net change between where you start and where you end is minus 2. So no matter what floor you start on, your net change is always going be -2.

Denise does not use an algebraic procedure to combine the function rules. Instead, she proceeds by combining the transformations and then writing a function for the resulting transformation, effectively bypassing the symbolic process of substituting the rule for the first function into the second function. It is also important to note that Denise's way of thinking appears to be deeply embedded in the elevator context. It seems that she is not reasoning about functions at all (at least explicitly), but is reasoning about changes in this context and then translating her results later into function notation.

Augmenting the First Function with the Second Change

Jim's first attempt involved adding the two function rules: $(x – 5) + (x + 2)$. He realized that he should not have $2x$ in his result and then reconsidered the task. He went on to develop an approach that was more symbolic than Denise's. However, like Denise, his reasoning was more about combining changes than combining functions.

Jim's Written Work

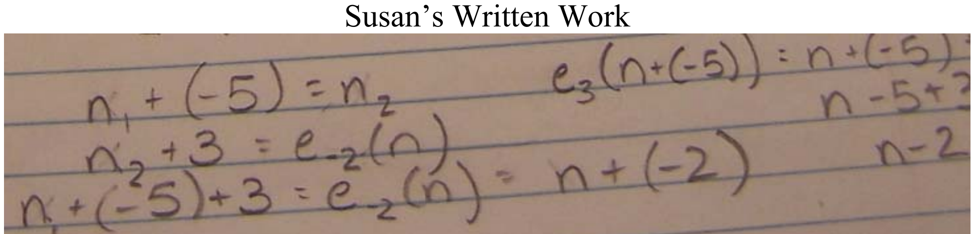


Jim combined the two functions, but not by substituting the output of one into the other. In response to the task of creating a function for the combination of the two buttons, he essentially created a new function from scratch, adding the first change and then the second to the starting floor x (see Figure above). In response to the task of describing a procedure for combining the two functions rules, Jim used a two-step process. He first thought of the rule for the first function, $x + (-5)$ as expressing the result of the first part of the trip, and then he considered the result of going up an additional 3 floors from this result, expressing symbolically as $(x + (-5)) + (+3)$. We note that this way of thinking is subtly different from substituting the output of one function into the other – it involves seeing the first function rule as expressing the result of the first part of the trip, and then applying the transformation associated with the second function rule to this result.

Using Notational Devices to Facilitate Substituting

An elementary teacher, Susan, introduced subscripts to deal with the fact that the starting floor for the second part of the trip was not the same as the starting point for the first part of the trip. This allowed her to write an expression, $n_1 + (-5) = n_2$, that expressed the fact that this second starting floor was the ending floor of the first part of the trip. She then was able to use this expression to make a substitution that resulted in the desired function rule. After doing so, she was able to do the substitution without using this notational device.

Susan's Written Work



Discussion Part 2: Implications for a Revised Local Instructional Theory

The approaches that Jim and Susan developed to deal with the tasks are especially relevant for our revision of the local instructional theory. Our data suggests that these two students were not drawing on prior procedural understanding of function composition – in fact, the evidence suggests that they were not aware this was function composition until a later juncture in the instructional sequence when this term was introduced. Note that, like Jim, Susan first attempted to combine the two functions by adding their rules.

Since Jim and Susan's reasoning was apparently based on the elevator context, we can learn something about how the symbolic procedure of substituting can emerge for students as they work in this context. From Jim's response, we see that it is possible to develop a symbolic procedure in this case that does not explicitly involve substitution. Because the functions here are of the form, $f(x) = x + b$, it is possible to easily compose them without substitution. One merely augments one of the functions by adding the change part (the "$+b$") of the other function. This suggests that it may be useful to introduce an additional function type that does not allow this approach.

From Susan's response we see that it may be difficult to think of the rule for the first function as an input for the second function. Note that a student needs to realize that this expression can stand for a floor (the ending floor) and not just a transformation associated with the function. They then need to combine this realization with a construal of this ending floor as the starting floor for the second part of the trip. We contend that construing this duality (Gray and Tall, 1994) entails significant conceptual complexity and coordination. Susan was able to introduce a notational device to help her manage this complexity. The use of subscripts allowed her to encapsulate the rule for the first function and think of it as an input of the second function. Later she was able to set this notational device aside and perform the substitution in the more standard way.

Revisions to Preliminary Local Instruction Theory / Directions for Further Research

Our findings suggest a number of possible refinements of the instructional theory. First, we found that the teachers had difficulty setting aside their tendency to think of integers exclusively as positions, even in a context designed to support thinking of integers as transformations. This finding suggested a need to rethink the point of departure for the instructional sequence. In preparation for further research to develop our instructional theory, we have designed a computer micro-world that aims to ground students' thinking in the dynamic elevator context. In this micro-world, the dominant feature is the idea of change. The goal is that as a student works with this micro-world, it will become clear that the elevator buttons represent changes and that these can be combined to generate other changes.

After the teachers became comfortable thinking in terms of combining changes in the elevator context, we found that they were able to leverage their ideas in different ways to think about function composition. These different ways of thinking may suggest plausible signposts on the way to developing a rich understanding of function composition. Two particularly important ways of thinking are expressed by the approaches of Jim and Susan. Jim's approach makes it clear that students can resort to symbolic procedures other than substitution to compose the kind of functions associated with the change buttons in the elevator context. One possible approach would be to introduce multiplier buttons into the elevator context (e.g. the "×4" button would take the elevator 4 times as far from the 0 floor). This kind of button would give a function of the type, $f(x) = ax$. It is much more difficult to compose this kind of function with one of the type, $f(x) = x + b$, without performing a substitution. Susan's approach suggest that it is important to make sure that students can see the rule of one of these functions as also representing a floor – the ending floor of the first part of the trip and the starting floor of the second part of the trip. Her approach also suggests a way to assist students in handling the complexity involved with thinking flexibly about, and working with, the rule of a function when composing two functions.

Introducing a notational device to help a student think of this expression as a floor may support their ability to link the two processes and compose the two functions.

As we continue to work to develop the local instructional theory, the next step will be to conduct a series of teaching experiments (Steffe & Thompson, 2000) in order to more carefully elaborate a path by which students can develop the concept of function composition as a way to formalize the process of combining transformations.

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