MAT 271 – Calculus II
Conic Sections

Here we give geometric definitions of parabolas, ellipses and hyperbolas and derive their standard equations.

1 Conic sections in Cartesian coordinates

- **Parabola**: Is the set of points in a plane that are equidistant from a fixed point F (called the *focus*) and a fixed line (called the *directrix*).
- **Vertex** is the point halfway between the focus and the directrix lying on the parabola.
- **Axis** of the parabola is the line through the focus perpendicular to the directrix.

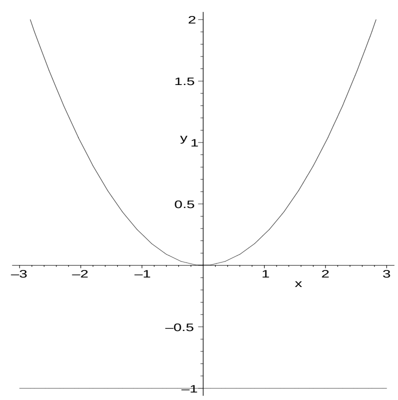

Figure 1: Parabola $x^2 = 4py$, $p > 0$.

Let us place the vertex at the origin O and its directrix parallel to the x-axis. The focus is the point $(0, p)$, then the directrix has the equation $y = -p$. If $P(x, y)$ is any point on the parabola, then the distance from P to the focus is

$$|PF| = \sqrt{x^2 + (y-p)^2} \tag{1}$$

and the distance from P to the directrix is $|y + p|$

The defining property of the parabola is that these distances are equal:

$$\sqrt{x^2 + (y-p)^2} = |y + p| \tag{2}$$

An equivalent expression is obtained by squaring and simplifying:

- The equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is:

$$x^2 = 4py \tag{3}$$

- It opens upward if $p > 0$.
- It opens downward if $p < 0$.

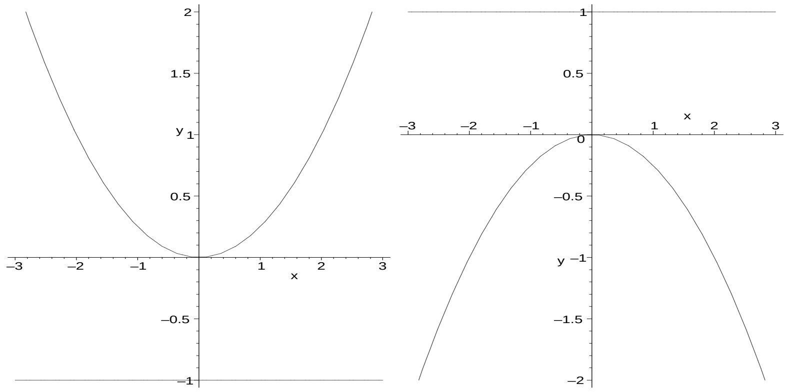

Figure 2: (a) $x^2 = 4py$, $p > 0$. (b) $x^2 = 4py$, $p < 0$.

- The equation of the parabola with focus $(p, 0)$ and directrix $x = -p$ is:

$$y^2 = 4px \tag{4}$$

- It opens to the right if $p > 0$.
- It opens to the left if $p < 0$.

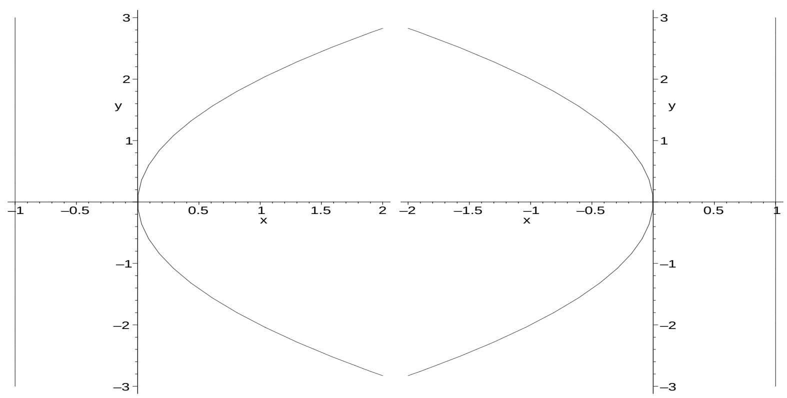

Figure 3: (a) $y^2 = 4px$, $p > 0$. (b) $y^2 = 4px$, $p < 0$.

- **Ellipse**: Is the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2 is a constant. The two fixed points F_1 and F_2 are called *foci*.

Let us place the foci on the x-axis at the points $(-c, 0)$ and $(c, 0)$ so that the origin is halfway between the foci. Let the sum of the distances from a point on the ellipse to the foci be $2a > 0$. Then the point $P(x, y)$ is a point on the ellipse when

$$|PF_1| + |PF_2| = 2a \tag{5}$$

that is

$$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a \tag{6}$$

which simplifies to

$$(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2) \tag{7}$$

For convenience, let $b^2=a^2-c^2$, then the equations of the ellipse becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{8}$$

The ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad a\geq b>0 \tag{9}$$

has foci $(\pm c,0)$ where $c^2=a^2-b^2$, and vertices $(\pm a,0)$.

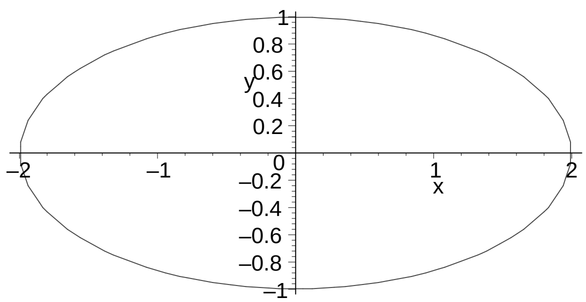

Figure 4: Foci $(\pm c,0)$.

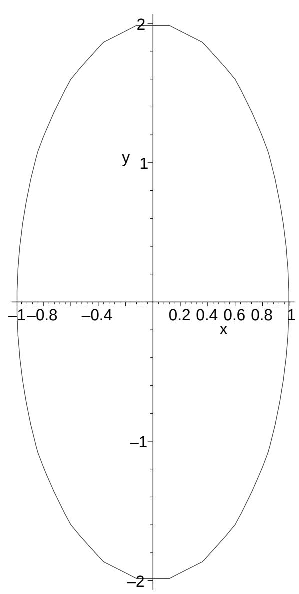

Figure 5: Foci $(0,\pm c)$.

The ellipse

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad a \geq b > 0 \tag{10}$$

has foci $(0, \pm c)$ where $c^2 = a^2 - b^2$, and vertices $(0, \pm a)$.

- **Hyperbola**: Is the set of points in a plane the difference of whose distances from two fixed points F_1 and F_2 is a constant. The two fixed points F_1 and F_2 are called *foci*.

Let us place the foci on the x-axis at the points $(-c, 0)$ and $(c, 0)$ so that the origin is halfway between the foci. Let the difference of the distances from a point on the hyperbola to the foci be $\pm 2a$. Then the point $P(x, y)$ is a point on the hyperbola when

$$|PF_1| - |PF_2| = \pm 2a \tag{11}$$

that is

$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = 2a \tag{12}$$

which simplifies to

$$(a^2 + c^2)x^2 - a^2y^2 = a^2(a^2 + c^2) \tag{13}$$

For convenience, let $c^2 = a^2 + b^2$, then the equations of the ellipse becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{14}$$

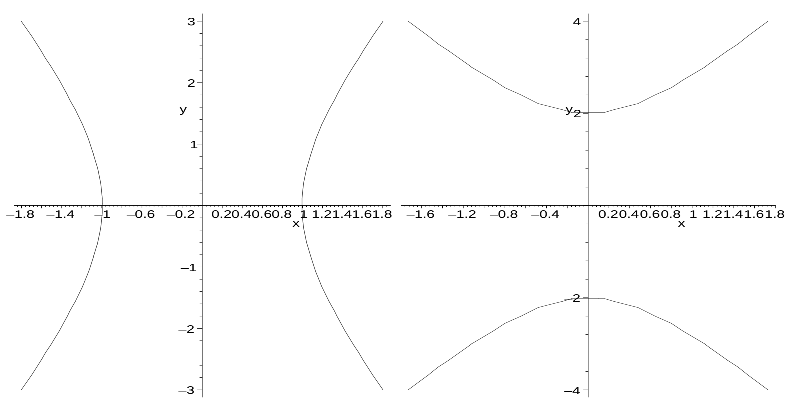

Figure 6: (a) Foci $(\pm c, 0)$. (b) Foci $(0, \pm c)$.

The hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad a \geq b > 0 \tag{15}$$

has foci $(\pm c, 0)$ where $c^2 = a^2 + b^2$, and vertices $(\pm a, 0)$, and asymptotes $y = \pm(b/a)x$.
The hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad a \geq b > 0 \tag{16}$$

has foci $(0, \pm c)$ where $c^2 = a^2 - b^2$, and vertices $(0, \pm a)$, and asymptotes $y = \pm(a/b)x$.

2 Conic sections in Polar coordinates

Let F be a fixed point called **focus** and l be a fixed line called the **directrix** in a plane. Let e be a fixed poisitive number called **eccenctricity**. The set of all points P in the plane such that

$$\frac{PF}{Pl} = e \tag{17}$$

(that is the ratio of the distance from F to the distance from l is the constant e) is a conic section. The conic is

1. an ellipse is $e < 1$
2. a parabola is $e = 1$
3. a hyperbola is $e > 1$

Place the focus F at the origin and the directrix parallel to the y-axis and d units to the right. Thus, the directrix has equation $x = d$ and is perpendicular to the polar axis. If the point P has polar coordinates (r, θ), we have

$$r = e(d - r\cos\theta) \tag{18}$$

or

$$x^2 + y^2 = e^2(d - x)^2 \tag{19}$$

After completing the square, we have

$$\left(x + \frac{d^2 d}{1 - e^2}\right) + \frac{y^2}{1 - e^2} = \frac{e^2 d^2}{(1 - e^2)^2} \tag{20}$$

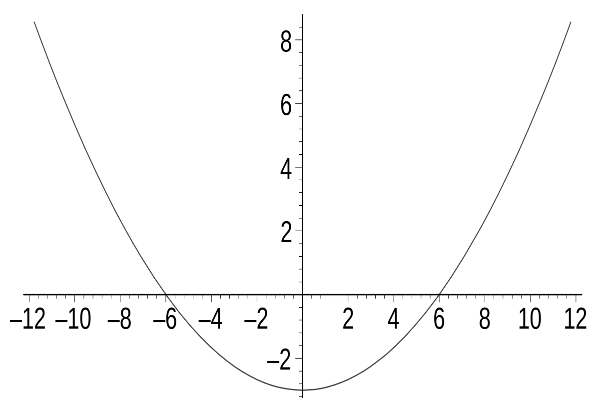

Figure 7: $r = \frac{6}{1 - \sin\theta}$

If $e < 1$ we have the ellipse, it is of the form:

$$\frac{(x - h)^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{21}$$

where

$$h = -\frac{e^2 d}{1 - e^2}, \qquad a^2 = \frac{e^2 d^2}{(1 - e^2)^2} \qquad b^2 = \frac{e^2 d^2}{1 - e^2}$$

where

$$c^2 = a^2 - b^2 = \frac{d^4 d^2}{(1-e^2)^2}$$

It also follows that the eccentricity is given by

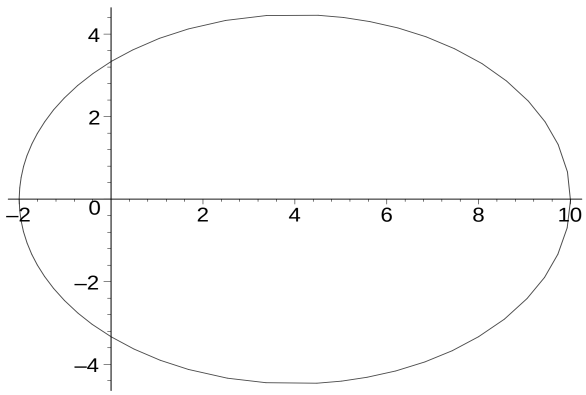

Figure 8: $r = \frac{10}{3-2\cos\theta}$

$$e = \frac{c}{a} \tag{22}$$

If $e > 1$ we have the hyperbola, it is of the form:

$$\frac{(x-h)^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{23}$$

and see that

$$e = \frac{c}{a} \qquad c^2 = a^2 + b^2$$

By solving (18) se see that the polar equation of the conic can be written as

$$r = \frac{ed}{1 + e\cos\theta} \tag{24}$$

A polar equation of the form

$$r == \frac{ed}{1 \pm e\cos\theta} \qquad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic section with eccentricity e. The conic is an ellipse if $e < 1$, a parabola if $e = 1$ or a hyperbola if $e > 1$.

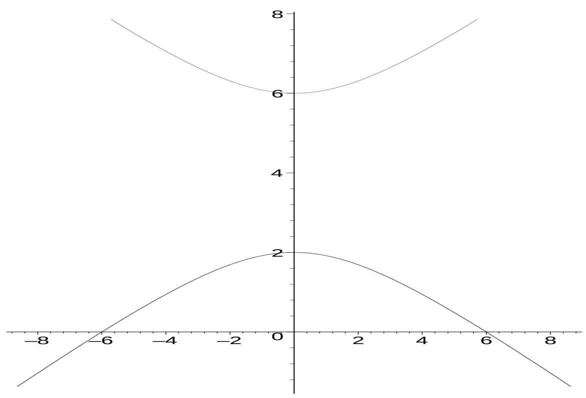

Figure 9: $r = \frac{6}{1+2\sin\theta}$