High Order Methods for Problems in Computational Aeroacoustics

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Abstract

This work concerns the numerical aspects of the direct simulation of acoustic wave propagation for problems in Computational Aeroacoustics (CAA). In CAA applications, efficient, high-order methods are needed to model large, high frequency waves in the near field, and small amplitude waves in the far field. The Chebyshev Pseudospectral Method (CPS) is a high order numerical method with phase and amplitude errors which decay exponentially as the number of points per wavelength increases. However, it is not very efficient. The phase and amplitude errors of variants of the CPS, which make CPS more efficient in large spatial approximations, are analyzed. For the temporal approximation, phase and amplitude errors of Runge kutta methods are analyzed. These numerical methods are applied to benchmark problems in CAA. A numerical method is formalized which balances high accuracy, minimal dispersion and dissipation, with the ability to approximate over large temporal and spatial scales efficiently.

1 Computational Aeroacoustics

In order to accurately predict a wave in the acoustic field, dispersion and dissipation must be minimal. Otherwise, the amplitude and frequency of the wave will be highly inaccurate because the errors are propagated over large distance and long time. Another consideration is the conflicting scales of the problem; in the far field the quantities modeled are potentially smaller than the computational error of low accuracy methods, whereas near the jet they will be large, [5].

Typically, aeroacoustic problems have been modeled with fourth or higher order accurate finite difference schemes, although some second order schemes have been used [2]. To increase the accuracy of the method it is not possible to continue increasing the order of the finite difference scheme because of the Runge phenomena associated with high-order equally-spaced interpolation on a finite interval. Moreover, wide stencils reduce flexibility near boundaries. Here we study, instead, pseudospectral methods which can provide high accuracy but potentially at a cost of expensive time-stepping.

2 Spectral Methods

Spectral methods, based on global interpolation to sets of data, offer superior accuracy to finite difference methods. Several formulations, and in particular Chebyshev collocation methods, are possible, but here we consider collocation or interpolatory methods for which the differential equation is satisfied exactly at the collocation points [1]. The derivative of

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function $u(x)$, approximated by $p_N(x)$, a polynomial of degree N, is obtained via

$$\frac{d}{dx}u(x) \approx \frac{d}{dx}p_N(x) = \sum_{j=0}^{N} u(x_j)\frac{d}{dx}l_j(x),$$

where the x_j are the Chebyshev points $x_j = \cos\frac{j\pi}{N}, \quad j = 0, 1, \ldots, N$. In vector notation, we write $u' = Du$ where D is the matrix differential operator with entries

$$D_{ij} = \frac{d}{dx}l_j(x)|_{x=x_i}.$$

One problem with the Chebyshev Pseudospectral method (CPS) is that for increasing N the eigenvalues of D are $O(N^2)$. Hence eigenvalue stability, which is weaker than asymptotic stability, predicts that time steps $O(1/N^2)$ must be taken. This makes computation over large time inefficient. To improve efficiency, a grid transformation can be applied, [4]. Various grid transformations have been investigated; the one presented in [4] was found to be most accurate. The Chebyshev collocation points x_i are transformed to $y_i = \frac{\sin^{-1}(\alpha x_i)}{\sin^{-1}(\alpha)}$. Thus the CPS differentiation matrix D is pre-multiplied by a diagonal matrix A with entries

$$A_{ii} = \frac{\sin^{-1}(\alpha)\sqrt{1-(\alpha x_i)^2}}{\alpha}.$$

As α approaches zero, the grid approaches the Chebyshev grid, and as α approaches 1, the grid becomes more equally spaced. This amounts to a pre-conditioning technique that reduces eigenvalues to $O(N)$. Theoretically larger time steps may be chosen and computation over large time is more efficient.

3 Accuracy of Pseudospectral Methods: Spatial

The phase and amplitude errors should be minimal in aeroacoustic applications. In [3] the phase and amplitude errors of the CPS method were determined numerically by solving the 1-D wave equation

(1) $\quad u_t + u_x = 0 \quad x \in [-1, 1], t \in (0, \infty)$

(2) $\quad u(x, 0) = exp(ik\pi x) \quad x \in (-1, 1] \quad u(-1, t) = exp(-k\pi(1+t)) \quad t \in (0, \infty).$

There are k waves in the interval for which the number of points per wavelength is given by $r = \frac{N}{k}$. Note that, as shown in [4], a minimum of π points per wavelength is needed for resolution with the CPS method, while 2 points per wavelength is needed for the Fourier method. However, this does not immediately provide measure of the phase and amplitude error even if the wave is resolved.

The analytic solution, $u(x, t) = exp(ik\pi(x - c_p t))$, has constant phase speed $c_p = 1$, while the computed solution takes the form $\hat{u}(x, t) = exp(i\theta_j)$. If we assume $\theta_j = a_j + ib_j$ is complex, then $\hat{u}(x, t) = e^{-b}(\cos a + i \sin a)$. The real part of the natural logarithm of the computed solution is the amplitude error and the imaginary part gives the phase speed error. As shown in [3] both the amplitude and phase error decay exponentially fast with the CPS method.

The preconditioned CPS method has been analyzed in the same way; phase and amplitude errors decay exponentially fast for small values of r, but around $r = 5$ we see that increasing the number of points per wavelength does not have a large impact on improving the solution. As the values of α approach 1, i.e., as the grid becomes equally spaced, solutions are not as accurate. If we look at the limiting case, $\alpha \to 1$,

$$A_{ii} = \frac{\pi}{2}\sqrt{1 - x_i^2}$$

has a maximum value of $\frac{\pi}{2}$ at $x = 0$, and is zero at $x = \pm 1$. Thus as α approaches one we are amplifying the value of the derivative at the center of the grid and damping out the endpoints. Since the boundary values are not zero, a more equally spaced grid gives less accurate solutions.

4 Time Discretization: Accuracy

For the numerical solution of Ordinary Differential Equations, it is common to do phase analysis with the test equation $\frac{du}{dt} = ibu$, b real. We will analyze the phase and amplitude errors of a fourth order Runge kutta (RK) method.

The computed solution of the test equation, with a fourth order RK method, is

$$(3)\quad u_{n+1} = (A_4(hb) + iB_4(hb))u_n, \quad A_4(hb) = 1 - \tfrac{(hb)^2}{2} + \tfrac{(hb)^4}{24}, \quad B_4(hb) = hb - \frac{(hb)^3}{6}.$$

While the analytical solution to the test equation is $u(t) = exp(ibt)u(0)$. The error will be minimal if $exp(ibh) \approx A_4(hb) + iB_4(hb)$. Similar to the phase and amplitude analysis in the spatial discretization, the amplitude error is given by $1 - \sqrt{A_4^2(hb) + B_4^2(hb)}$, while the phase error is given by $hb - \tan^{-1}\left(\frac{B_4(hb)}{A_4(hb)}\right)$, [6]. The phase and amplitude error was found by using the Taylor expansion of $\tan(hb)$, and substituting $A_4(hb)$ and $B_4(hb)$ from (3). It was found that the amplitude error of a fourth order RK method is zero up to order h^4, while the phase error is zero up to order h^6.

5 Conclusions

The pre-conditioned CPS method and a fourth order RK method were used to solve the spherical wave problem in [2]. Since the boundary was zero, accurate results were found in a reasonable amount of time. While the large spatial discretization, and long time integration in the problem, made application of the CPS method inefficient. However, for problems where the boundary values is not zero, some accuracy will be lost with the pre-conditioned CPS method. Our results will show that there is some trade-off between the CPS and the pre-conditioned CPS method.

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