ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 34, Number 1, Spring 2004

THREE-DESCENT AND THE BIRCH AND SWINNERTON-DYER CONJECTURE

ANDREA BANDINI

ABSTRACT. We give a three-descent procedure to bound and, in some cases, compute the three-part of the Selmer and Tate-Shafarevich group of the curves $y^2 = x^3 + a$, a a nonzero integer. This enables us to verify the whole Birch and Swinnerton-Dyer conjecture for some of such curves.

1. Introduction. Let E be an elliptic curve defined over $\mathbf{Q}$ with complex multiplication by the ring of integers O_F of a quadratic imaginary field F. Let $Ш(E/\mathbf{Q})$ be the Tate-Shafarevich group of E over $\mathbf{Q}$. The Birch and Swinnerton-Dyer conjecture (first formulation [**1**], 1965, refined in [**5**], 1982) relates the rank of $E(\mathbf{Q})$ and the order of $Ш(E/\mathbf{Q})$ to the behavior of a certain L-function, associated with E, at 1.

The first major step towards a proof of the conjecture was made by Coates and Wiles in 1977 in two papers ([**3, 4**]) in which they proved that for an elliptic curve E with complex multiplication

$$\operatorname{rank} E(\mathbf{Q}) \geq 1 \Longrightarrow L(E_{/\mathbf{Q}}, 1) = 0.$$

Later Rubin in a series of papers ([**7, 8** and many others]) proved, among other important results, that if $\operatorname{rank} E(\mathbf{Q}) = 0$ then the conjecture holds up to primes dividing $\#O_F^*$.

In this paper we deal with curves $E_a : y^2 = x^3 + a$ with $a \in \mathbf{Z} - \{0\}$ which admit complex multiplication by the ring of integers of $\mathbf{Q}(\sqrt{-3})$. This is the only case in which 3 divides $\#O_F^*$ and our goal is to bound or, in some cases, compute exactly the order of the three-part of the Tate-Shafarevich group of such curves.

In Section 2 we shall give precise definitions for the groups we are interested in and fix notations for the rest of the paper.

Received by the editors on February 9, 2001, and in revised form on June 14, 2001.