MAT 571 Real Analysis II

John Quigg Spring 2009

revised April 3, 2009

Notes on: Proposition 7.19

You'll recall that yesterday (4209) we were trying to prove Prop.7.19 part (b), and I got hung up on the following: given $\varepsilon > 0$, we noticed that there exists $b < a$ such that $F(b) < \varepsilon$, and I wanted to show that there exists n_0 such that for all $n \geq n_0$ we have $F_n(b) < \varepsilon$. Below I'll give the argument (which I'll also discuss on Tuesday), using Stephen's idea. Actually, to keep the notation clean, the argument will in fact give $F_n(b) < 4\varepsilon$, which of course could then be adjusted to ε by using $\varepsilon/4$ throughout the argument.

First choose (using regularity of ν) $f \in C_0(\mathbb{R})$ such that

$$0 \leq f \leq \chi_{(b,\infty)}$$

and

$$\int f\,d\nu > \nu(b,\infty) - \varepsilon.$$

Next choose n_0 such that for all $n \geq n_0$ we have both

$$\int f\,d\nu_n > \int f\,d\nu - \varepsilon$$

and

$$\|\nu_n\| < \|\nu\| + \varepsilon,$$

and so

$$\begin{aligned}
F_n(b) &= \nu_n(-\infty, b] \\
&= \nu_n(\mathbb{R}) - \nu_n(b,\infty) \\
&= \|\nu_n\| - \nu_n(b,\infty) \\
&< \|\nu\| + \varepsilon - \int f\,d\nu_n \qquad \left(\text{because } \int f\,d\nu_n \leq \nu_n(b,\infty)\right) \\
&< \|\nu\| - \int f\,d\nu + 2\varepsilon \\
&< \|\nu\| - \nu(b,\infty) + 3\varepsilon \\
&= \nu(-\infty, b] + 3\varepsilon \\
&< 4\varepsilon.
\end{aligned}$$