

# MAT 473 Intermediate Real Analysis II

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## Extended reals

We will now switch abruptly from derivatives to integrals — Lebesgue integrals, that is. If you have only studied Riemann integrals, it will take a while to get used to the Lebesgue ones. The fundamental — and far-reaching — difference is that Riemann integrals chop up the domain of the function and look at the values of the function on each bit, while Lebesgue integrals chop up the range of the function and look at the sets where the function takes values in each bit. By “look at” in the case of Lebesgue integrals, we mean “measure the size”. This leads to a perhaps surprisingly extensive investigation into how to measure sets.

One crippling weakness of the Riemann integral is that it does not behave very well with respect to limits of sequences of functions — to get anything reasonable we need a very strong hypothesis: uniform convergence. In contrast, the Lebesgue integral behaves extremely well with respect to limits of sequences; in fact, the construction of the Lebesgue integral is designed with precisely this purpose in mind. The basic idea is to allow all *countable* operations.

We have to come to grips straight away with one consequence of the desire to allow unfettered countable operations: it is convenient, and customary, to use plus and minus infinity in a technical sense:

### Definition 1.

- (1) The set of *extended real numbers* is defined as

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\},$$

where  $+\infty$  and  $-\infty$  are distinct objects which are not real numbers.

- (2) We write  $\infty$  for  $+\infty$ .

- (3) The ordering of  $\mathbb{R}$  is extended to  $\overline{\mathbb{R}}$  by

$$-\infty < x < \infty \quad \text{for } x \in \mathbb{R}.$$

- (4) Terms such as maximum or minimum, upper or lower bound, and supremum or infimum are applied to sets of extended real numbers in the obvious way.

**Lemma 2.** *If  $A \subset \overline{\mathbb{R}}$  is nonempty, then:*

- (1)  $A$  has a sup;  
(2) If  $A \subset \mathbb{R}$ , then  $A$  is bounded above in  $\mathbb{R}$  if and only if  $\sup A < \infty$ .

*Similarly for inf.*

**Definition 3.** The *limsup* of a sequence  $(x_n)$  in  $\overline{\mathbb{R}}$  is

$$\limsup x_n := \inf_{k \in \mathbb{N}} \sup_{n \geq k} x_n.$$

Similarly for *liminf*.

**Corollary 4.** *Every sequence in  $\overline{\mathbb{R}}$  has both a lim sup and a lim inf.*

**Notation and Terminology.** If  $(x_n)$  is a sequence in  $\overline{\mathbb{R}}$  and  $x \in \overline{\mathbb{R}}$ , then “ $x_n \rightarrow x$  in  $\overline{\mathbb{R}}$ ” means  $x = \limsup x_n = \liminf x_n$ . We also write  $x = \lim x_n$ .

**Corollary 5.** *Every monotone sequence in  $\overline{\mathbb{R}}$  has a limit in  $\overline{\mathbb{R}}$ .*

We will need the particular case of increasing limits often enough to warrant taking advantage of the following, fairly standard, notation:

**Notation and Terminology.**  $x_n \uparrow x$  means  $(x_n)$  is an increasing sequence in  $\overline{\mathbb{R}}$  and  $x_n \rightarrow x$  in  $\overline{\mathbb{R}}$ . Similarly for  $x_n \downarrow x$ .

We need a limited sort of arithmetic in the extended real numbers:

**Definition 6.**

- (1) If  $x > -\infty$  then  $x + \infty = \infty + x = \infty$ .
- (2) If  $x < \infty$  then  $x - \infty = -\infty + x = -\infty$ .
- (3) If  $x > 0$  then  $x(\pm\infty) = (\pm\infty)x = \pm\infty$ .
- (4) If  $x < 0$  then  $x(\pm\infty) = (\pm\infty)x = \mp\infty$ .
- (5)  $0(\pm\infty) = (\pm\infty)0 = 0$ .

Thus if  $a_n \geq 0$  then the series  $\sum_{n=1}^{\infty} a_n$  always has a sum in  $[0, \infty]$ . A special note concerning (5) above: you were admonished in your calculus course that zero times infinity is undefined (and in the context of limits it is called an “indeterminate form”); but allowing zero times infinity to be zero turns out to be convenient *for very special purposes* in the Lebesgue integration theory — we just have to be careful how we use it.