MAT 473 Intermediate Real Analysis II

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Change of variables — Exercises

1. Let A be the region in the first quadrant of $\mathbb{R}^2$ bounded by the curves $xy = 1$, $xy = 3$, $x^2 - y^2 = 1$, and $x^2 - y^2 = 4$. Find

$$\int_A (x^2 + y^2)\, dxdy$$

by making the substitution $u = x^2 - y^2$ and $v = xy$. Be sure to fully explain how you are applying the Change of Variables Theorem, including checking the hypotheses.

2. Show that:

   (a) the volume of the region

$$\left\{ x \in \mathbb{R}^n \;\middle|\; x_1, \ldots, x_n \geq 0, \sum_1^n x_i \leq 1 \right\}$$

   is

$$\frac{1}{n!}$$

   (b) if $v_1, \ldots, v_n \in \mathbb{R}^n$ are linearly independent, then the volume of the region

$$\left\{ x_1 v_1 + \cdots + x_n v_n \;\middle|\; x_1, \ldots, x_n \geq 0, \sum_{i=1}^n x_i \leq 1 \right\}$$

   is

$$\frac{\left|\det \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}\right|}{n!}$$

3. Let f be a measurable function on $\mathbb{R}$. Prove that the function $(x, y) \mapsto f(x - y)$ on $\mathbb{R}^2$ is measurable.

4. Let f and g be integrable functions on $\mathbb{R}$. Prove that the function $y \mapsto f(x - y)g(y)$ is integrable for a.e. $x \in \mathbb{R}$, and that the a.e.-defined function $f * g : \mathbb{R} \to \mathbb{R}$ defined by

$$f * g(x) := \int_{-\infty}^{\infty} f(x - y)g(y)\, dy$$

   is integrable.

   Hint: show that the function $h(x, y) := f(x-y)g(y)$ is integrable on $\mathbb{R}^2$ by applying Tonelli's Theorem to $|h|$, then apply Fubini's Theorem to h.

5. *Cylindrical Coordinates.* Define $g : \mathbb{R}^3 \to \mathbb{R}^3$ by
$$g(r, \theta, z) = (r\cos\theta, r\sin\theta, z).$$
Similarly to the example in the lecture notes, find appropriate sets A, U, V if B is to be $\mathbb{R}^3$, and find the corresponding change-of-variables formula for integrals.

6. *Spherical Coordinates.* Define $g : \mathbb{R}^3 \to \mathbb{R}^3$ by
$$g(\rho, \phi, \theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi).$$
Similarly to the above exercise concerning cylindrical coordinates, find appropriate sets A, U, V if B is to be $\mathbb{R}^3$, and find the corresponding change-of-variables formula for integrals.

7. Use cylindrical coordinates to find the volume of the solid torus obtained by revolving the disk
$$(x-a)^2 + z^2 \leq b^2$$
in the xz-plane around the z-axis, where $0 < b < a$.

8. Use polar coordinates to integrate $e^{-x^2-y^2}$ over $\mathbb{R}^2$, and use this to help show that
$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$