MAT 473 Intermediate Real Analysis II

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Iterated integrals — Exercises

1. Let f be a measurable function on X, and define g on $X \times Y$ by $g(x, y) = f(x)$. Prove that g is measurable. Hint: use the definition of measurable function.

2. Define f on $(0, 1) \times (0, 1)$ by
$$f(x, y) = \begin{cases} 1/x^2 & \text{if } y < x < 1 \\ -1/y^2 & \text{if } x < y < 1. \end{cases}$$
Show:

   (a) $\int_0^1 \int_0^1 f(x, y)\, dy\, dx = 1$;

   (b) $\int_0^1 \int_0^1 f(x, y)\, dx\, dy = -1$.

3. Define $f(x, y) = 1/x^3$ if $0 < y < |x| < 1$ and 0 otherwise. Show:

   (a) $\int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y)\, dx\, dy = 0$ and

   (b) $\int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y)\, dy\, dx$ does not exist.

4. Let $f : \mathbb{R} \to [0, \infty)$ be measurable, and put
$$G = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le f(x)\}.$$
Prove that G is measurable and
$$m(G) = \int f.$$
Hint: for the first part, show that the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x, y) = f(x) - y$ is measurable. For the second part, use Tonelli's Theorem to integrate with respect to y first.

5. Define $f(x, y) = xy/(x^2 + y^2)^2$ away from the origin, and let $f(0, 0) = 0$. Prove:

   (a) $\int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y)\, dx\, dy = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y)\, dy\, dx = 0$, but

   (b) f is not integrable.

6. Let f be integrable on $(0,a)$ (meaning that f is measurable on $(0,a)$, and the trivial extension of f to $\mathbb{R}$, with constant value 0 outside $(0,a)$, is integrable). Prove that
$$\int_0^a \int_y^a \frac{f(x)}{x}\,dx\,dy = \int_0^a f.$$
Hint: use Tonelli's Theorem to prove that the function $(x,y) \mapsto f(x)/x$ is integrable on the set
$$A := \{(x,y) : 0 \le y \le a, y \le x \le a\},$$
then use Fubini's Theorem to interchange the order of integration.

7. In this exercise you'll show that
$$\lim_{b\to\infty} \int_0^b \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$
Let $f(x,y) = e^{-xy}\sin x$ and $A = (0,b)\times(0,\infty)$ (where without loss of generality $b > 0$).

   (a) Prove that f is integrable on A by *carefully* estimating $|f|$.

   (b) Integrate f over A in one order to get $\int_0^b \sin x/x\,dx$.

   (c) Integrate f over A in the other order to get
$$\frac{\pi}{2} - \sin b \int_0^\infty \frac{ye^{-by}}{1+y^2}\,dy - \cos b \int_0^\infty \frac{e^{-by}}{1+y^2}\,dy.$$

   (d) Show how the Dominated Convergence Theorem can be used to help prove that both of the latter integrals go to 0 as $b \to \infty$.

   (e) Indicate how this implies the desired result.

8. Let $0 < a < b < \infty$. Integrate e^{-xy} over $A := (0,\infty)\times(a,b)$ in two different ways to help show that
$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx = \log\frac{b}{a}.$$

9. Prove that if $f : \mathbb{R}^n \to [0,\infty]$ is locally integrable and $A \in \mathcal{L}$ then
$$\int_A f = \sup\left\{\int_K f : K \text{ is compact and } K \subset A\right\}.$$
Hint: mimic the proofs of Corollaries 10 and 11 in Section 8.