MAT 473 Intermediate Real Analysis II

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Integrable functions — Exercises

1. (Translation-Invariance of Lebesgue Integration) Let $c \in \mathbb{R}$. Prove that if $f \in L^1(\mathbb{R})$, then so is $x \mapsto f(x+c)$, and
$$\int f(x+c)\,dx = \int f.$$

2. Prove that there is no continuous function equal to $\chi_{[0,1]}$ a.e. on the interval $[0,2]$.

3. (Dominated Convergence Theorem for Continuous Limits) Let T be a metric space, let $A \subset T$, let s be a cluster point of A, let (X,μ) be a measure space, and let $f : X \times A \to \mathbb{R}$. Assume that
$$x \mapsto f(x,t) \text{ is integrable for every } t \in A,$$
that
$$\lim_{t\to x} f(x,t) \text{ exists for every } x \in X,$$
and that there exists $g \in L^1(X,\mu)$ such that
$$|f(x,t)| \le g(x) \text{ for all } (x,t) \in X \times A.$$
Prove that the function
$$x \mapsto \lim_{t\to s} f(x,t)$$
is integrable, that
$$\lim_{t\to s} \int f(x,t)\,d\mu = \int \lim_{t\to s} f(x,t)\,f\,d\mu,$$
and that as t goes to s the functions $x \mapsto f(x,t)$ go to the function $x \mapsto \lim_{t\to s} f(x,t)$ in $L^1(X,\mu)$.

   Hint: consider an arbitrary sequence $\{t_n\}$ in $A \setminus \{s\}$ which converges to s.

4. (Differentiation under the Integral) Let $f : \mathbb{R} \times (a,b) \to \mathbb{R}$. Assume that
$$x \mapsto f(x,y) \text{ is integrable for every } y \in (a,b),$$
that
$$\frac{\partial f}{\partial y}(x,y) \text{ exists for all } (x,y) \in \mathbb{R} \times (a,b),$$
and that there exists $g \in L^1(\mathbb{R})$ such that
$$\left|\frac{\partial f}{\partial y}(x,y)\right| \le g(x) \qquad \text{for all } (x,y) \in \mathbb{R} \times (a,b).$$
Prove that the function
$$y \mapsto \int f(x,y)\,dx$$

is differentiable on (a, b), and

$$\frac{d}{dy}\int f(x,y)\,dx = \int \frac{\partial f}{\partial y}(x,y)\,dx.$$

5. Find and justify:

$$\lim_{k\to\infty}\int_0^\infty \frac{k\sin(x/k)}{x(1+x^2)}\,dx.$$

6. Let $f \in L^1$, and let $A_1, A_2, \ldots \in \mathcal{M}$ be disjoint. Prove that

$$\int_{\bigcup_1^\infty A_n} f = \sum_1^\infty \int_{A_n} f.$$

7. Prove that if $\chi_{A_n} \to f$ in L^1, then f is a.e. equal to a characteristic function.

8. (Absolute Continuity) Let $f \in L^1$. Prove that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|\int_A f\right| < \varepsilon \qquad \text{whenever } \mu(A) < \delta.$$

Hint: consider a sequence (A_n) in $\mathcal{M}$ with $\mu(A_n) \to 0$.

9. Let $f \in L^1(\mathbb{R})$, and suppose $\int_0^t f = 0$ for all $t \in \mathbb{R}$. Prove that $f = 0$ almost everywhere. Hint: sets of finite measure can be approximated by finite unions of bounded intervals.

10. (Continuity of Translation) Let $f \in L^1(\mathbb{R})$. For each $t \in \mathbb{R}$ define f_t by

$$f_t(x) = f(x+t).$$

Also let $s \in \mathbb{R}$. In this problem you'll follow a prescribed strategy to prove that

$$\lim_{t\to s} f_t = f_s \qquad \text{in } L^1. \tag{1}$$

(a) Use the Dominated Convergence Theorem to prove (1) for the case $f \in C_c(\mathbb{R})$. Hint: when letting $t \to s$, it suffices to consider t in the interval $(s-1, s+1)$.

(b) Use part (a) and density of $C_c(\mathbb{R})$ in L^1 to deduce (1) in general. Hint: for $t \in \mathbb{R}$, the map $f \mapsto f_t$ is a linear operator on $L^1(\mathbb{R})$ such that, by translation invariance of Lebesgue integration, $\|f_t\|_1 = \|f\|_1$ for all $f \in L^1(\mathbb{R})$.

11. (Riemann-Lebesgue Lemma) For each $f \in L^1(\mathbb{R})$ define $\Psi(f) : \mathbb{R} \to \mathbb{R}$ by

$$\Psi(f)(t) = \int_{-\infty}^\infty f(x)\sin(xt)\,dx.$$

(a) Prove that if $f_n \to f$ in L^1 then $\Psi(f_n) \to \Psi(f)$ uniformly on $\mathbb{R}$.

(b) Prove that for every $f \in L^1(\mathbb{R})$,

$$\lim_{t\to\infty} \Psi(f)(t) = 0.$$

Hint: first take f to be the characteristic function of a bounded interval. You may use elementary integration formulas from calculus.

12. (a) Is the characteristic function of the Cantor set Riemann integrable?

    (b) Is the characteristic function of a fat Cantor set Riemann integrable?

    Give full justifications.