MAT 473 Intermediate Real Analysis II

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revised January 14, 2009

Higher order derivatives — Exercises

1. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by
$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$
Show that $\dfrac{\partial^2 f}{\partial x \partial y}(0,0)$ and $\dfrac{\partial^2 f}{\partial y \partial x}(0,0)$ both exist, but are not equal.

2. Let $f, g : \mathbb{R} \to \mathbb{R}$ be C^2. Prove that $v(x,y) = f(x-y) + g(x+y)$ satisfies the *wave equation*
$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = 0.$$

3. Let $f, g, h : \mathbb{R}^2 \to \mathbb{R}$ and $F : \mathbb{R}^2 \to \mathbb{R}$ be C^2. Put
$$w = F(x,y),$$
where
$$x = f(u,v) \quad \text{and} \quad y = g(u,v).$$
Show that
$$\begin{aligned} \frac{\partial^2 w}{\partial u^2} = \frac{\partial^2 w}{\partial x^2}\left(\frac{\partial x}{\partial u}\right)^2 &+ \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial y}{\partial u}\right)^2 \\ &+ 2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial x}{\partial u}\frac{\partial y}{\partial u} \\ &+ \frac{\partial w}{\partial x}\frac{\partial^2 x}{\partial u^2} + \frac{\partial w}{\partial y}\frac{\partial^2 y}{\partial u^2}. \end{aligned}$$

4. Let $U \subset \mathbb{R}^n$ be open, and let $f : U \to \mathbb{R}$ be C^2. The *Hessian* of f at $y \in U$ is the $n \times n$ matrix
$$H(y) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(y)\right).$$

   (a) Prove that
$$\sum_{|\alpha|=2} \frac{1}{\alpha!}\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(y) x^\alpha = \frac{1}{2} x^t H(y) x.$$
   Hint: $H(y)$ is symmetric by Clairaut's Theorem.

(b) Prove that for all $a \in U$ there exists $\delta > 0$ such that for all $x \in B_\delta(0)$ there exists $t \in (0,1)$ such that

$$f(a+x) = f(a) + f'(a)x + \frac{1}{2}x^t H(a+tx)x.$$

5. Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ be C^2, and $a \in U$. Let H be the Hessian of f at a. Prove that

$$\lim_{x \to 0} \frac{f(a+x) - f(a) - f'(a)x - \frac{1}{2}x^t Hx}{\|x\|^2} = 0.$$

6. Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ be C^2, and $a \in U$. Let $H = H(a)$ be the Hessian of f at a.

   (a) Prove that if H is positive definite then there exists $\delta > 0$ such that for all $y \in B_\delta(a)$ the Hessian $H(y)$ is positive definite.

   (b) State and prove an analogous result if H is negative definite.

7. (Second Derivative Test) Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ be C^2, and $a \in U$. Let $H = H(a)$ be the Hessian of f at a. Assume that $f'(a) = 0$.

   (a) Prove that if H is positive definite, then there exists $\delta > 0$ such that if $0 < \|x\| < \delta$ then $f(a+x) > f(a)$.

   (b) State and prove an analogous result if H is negative definite.

   (c) Suppose that there exist $u, v \in \mathbb{R}^n$ such that $u^t Hu > 0$ and $v^t Hv < 0$. Prove that for all $\gamma > 0$ there exist $x, y \in B_\gamma(0)$ such that

$$f(a+x) > f(a) > f(a+y).$$

8. Suppose U is an open subset of $\mathbb{R}^2$, $(a,b) \in U$, and $f : U \to \mathbb{R}$ is C^3. Prove that

$$\lim_{r \to 0} \frac{4}{\pi r^2} \int_0^{2\pi} f(a + r\cos\theta, b + r\sin\theta)\cos 2\theta \, d\theta = \frac{\partial^2 f}{\partial x^2}(a,b) - \frac{\partial^2 f}{\partial y^2}(a,b).$$

   Hint: use Taylor's Theorem with $m = 2$. Also, you may use the following following standard integrals from calculus:

$$\begin{aligned}\int_0^{2\pi} \cos 2\theta \, d\theta &= \int_0^{2\pi} \cos\theta\cos 2\theta \, d\theta = \int_0^{2\pi} \sin\theta\cos 2\theta \, d\theta \\ &= \int_0^{2\pi} \cos\theta\sin\theta\cos 2\theta \, d\theta = 0\end{aligned}$$

$$\int_0^{2\pi} \cos^2\theta\cos 2\theta \, d\theta = \frac{\pi}{2}, \qquad \int_0^{2\pi} \sin^2\theta\cos 2\theta \, d\theta = -\frac{\pi}{2}$$