MAT 473 Intermediate Real Analysis II

John Quigg

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Inverse functions — Exercises

1. Let $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}^n$, and assume that f is differentiable at $a \in E$. Also assume that f is 1-1, so that we have an inverse function $f^{-1} : f(E) \to E$. Finally, assume that f^{-1} is differentiable at $f(a)$.

   (a) Prove that $f'(a)$ is invertible and
$$(f^{-1})'(f(a)) = f'(a)^{-1}.$$

   (b) Carefully explain how part (a) implies that if f' is continuous at a then $(f^{-1})'$ is continuous at $f(a)$.

2. Define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} x + 2x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$
Show that f is differentiable on $\mathbb{R}$ and $f'(0) \neq 0$, but f is not 1-1 on any open interval containing 0.

3. Let $U = \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$, and define $f : U \to \mathbb{R}^2$ by
$$f(x, y) = \left(e^x + xy^2, \frac{2 \sin \pi x}{y}\right).$$
Note that $f(1, 1) = (e + 1, 0)$.

   (a) Use the Inverse Function Theorem to show that f is invertible near $(1, 1)$, and find a formula for (the matrix representing) $(f^{-})'(e + 1, 0)$.

   (b) Why does the Inverse Function Theorem not apply to the question of whether f is invertible near $(0, 1)$?

4. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by
$$f(x, y) = \big(e^x \cos y, e^x \sin y\big).$$
Prove that f is C^1, and $f'(x, y)$ is invertible for all $(x, y) \in \mathbb{R}^2$, but f is not 1-1.