MAT 473 Intermediate Real Analysis II

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Derivatives — Exercises

1. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = \sqrt{|xy|}$. Is f differentiable at 0?

2. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $a > e1$. Suppose that
$$\|f(x)\| \le \|x\|^a \qquad \text{for all } x \in \mathbb{R}^n.$$

   (a) Prove that if $a > 1$ then f is differentiable at 0, with $f'(0) = 0$.

   (b) What if $a = 1$?

3. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = 3xy$. Find $f'(2, 1)(3, 2)$.

4. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by
$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \ne (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$
Show that at $(0, 0)$, both partial derivatives of f exist, but f is not even continuous.

5. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by
$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{if } (x,y) \ne (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$
Show that at $(0, 0)$, f is continuous and both partial derivatives exist, but f is not differentiable at $(0, 0)$.

6. (Directional Derivative) Let $E \subset \mathbb{R}^n$, $f : E \to \mathbb{R}$, $a \in E^\circ$, and $u \in \mathbb{R}^n$. The *directional derivative* of f at a in the direction u is
$$D_u f(a) := \lim_{t \to 0} \frac{f(a + tu) - f(a)}{t} = \frac{d}{dt} f(a + tu) \bigg|_{t=0}$$
Prove that if f is differentiable at a then
$$f'(a)u = D_u f(a).$$

7. Let
$$f(x,y) = \begin{cases} \dfrac{x^2y\sqrt{x^2+y^2}}{x^4+y^2} & \text{if } (x,y) \ne (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Prove that all the directional derivatives of f at $(0,0)$ are 0.

(b) Prove that f is nondifferentiable at $(0,0)$. Hint: try going to $(0,0)$ along a parabola.

8. (Gradient) Let $E \subset \mathbb{R}^n$ be open, $f : E \to \mathbb{R}$, and $a \in E^\circ$. The *gradient vector* of f at a is
$$\nabla f(a) := \left(\frac{\partial f}{\partial x_1}(a), \dots, \frac{\partial f}{\partial x_n}(a) \right).$$
Suppose that f is differentiable at a. Obviously, for $x \in \mathbb{R}^n$ we have
$$f'(a)x = \nabla f(a) \cdot x.$$
Prove that for fixed a, among all unit vectors $u \in \mathbb{R}^n$, the maximum directional derivative $D_u f(a)$ is obtained when
$$u = \frac{\nabla f(a)}{\|\nabla f(a)\|},$$
and moreover this maximum directional derivative is $\|\nabla f(a)\|$.

9. Prove that if $r \in \mathbb{R}$ then on $\mathbb{R}^n$ we have
$$\nabla \|x\|^r = r\|x\|^{r-2}x.$$

10. (Tangent Vector) A *simple smooth curve* in $\mathbb{R}^n$ is a set $C \subset \mathbb{R}^n$ which is the range of a 1-1 differentiable function $g : [a,b] \to \mathbb{R}^n$, where $[a,b]$ is a compact interval in $\mathbb{R}$ and $g'(t) \neq 0$ for all $t \in [a,b]$. A linear map $T : \mathbb{R} \to \mathbb{R}^n$ can be identified with the vector $v = T(1) \in \mathbb{R}^n$, so that $T(x) = xv$ for $x \in \mathbb{R}$. In this way, for $t_0 \in [a,b]$ we identify the derivative $g'(t_0)$ with a vector, called a *tangent vector* to the curve C at the point $g(t_0)$. Prove that as a vector in $\mathbb{R}^n$ we have
$$g'(t_0) = \lim_{t \to t_0} \frac{g(t) - g(t_0)}{t - t_0}.$$

11. Of course, the tangent vector from the preceding problem depends upon the *parameterization* g of the curve. However, in fact it's really only the length and the "orientation" of the vector $g'(t_0)$ that depend upon g. More precisely, let $\phi : [c,d] \to [a,b]$ be a strictly increasing differentiable function with $\phi'(s) \neq 0$ for all $s \in [c,d]$, so that $h = g \circ \phi$ is another parameterization of the curve C. Let $s_0 \in [c,d]$ and $t_0 = \phi(s_0)$. Prove that the tangent vector $h'(s_0)$ is a positive multiple of the tangent vector $g'(t_0)$, while on the other hand if ϕ is strictly decreasing then $h'(s_0)$ is a negative multiple of $g'(t_0)$.

12. Let $D \subset \mathbb{R}$, $E \subset \mathbb{R}^n$, $g : D \to E$, $f : E \to \mathbb{R}$, $a \in D$, and $b = g(a)$. Suppose g is differentiable at a and f is differentiable at b. Prove that
$$(f \circ g)'(a) = \nabla f(b) \cdot g'(a) = \sum_1^n \frac{\partial f}{\partial x_i}(b) g_i'(a).$$

13. (Normal Vector) Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable, and let $c \in \mathbb{R}^n$. The *level set* of f through c is
$$S = \{x \in \mathbb{R}^n : f(x) = f(c)\}.$$
Prove that the gradient vector of f at c is *normal* to S in the sense that if C is any simple smooth curve which is contained in S and contains the point c then $\nabla f(c)$ is orthogonal to every tangent vector to C at c.

14. Let $f : \mathbb{R} \to \mathbb{R}^n$ be differentiable, and suppose that $\|f\|$ is constant. Prove that for all $t \in \mathbb{R}$ the vectors $f(t)$ and $f'(t)$ are orthogonal.

15. Let $D \subset \mathbb{R}^k$, $E \subset \mathbb{R}^n$, $g : D \to E$, $f : E \to \mathbb{R}$, $a \in D$, and $b = g(a)$. Suppose g is differentiable at a and f is differentiable at b. Prove that for all $i\jmath = 1, \ldots, k$ we have
$$\frac{\partial (f \circ g)}{\partial u_j}(a) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(b) \frac{\partial g_i}{\partial u_j}(a).$$

16. Let $f, g, h : \mathbb{R}^2 \to \mathbb{R}$ and $F : \mathbb{R}^3 \to \mathbb{R}$ be differentiable. Put
$$w = F(x, y, z),$$
where
$$x = f(u, v), \quad y = g(u, v), \quad \text{and} \quad z = h(u, v).$$
Find formulas for
   (a) $\dfrac{\partial w}{\partial u}$ and
   (b) $\dfrac{\partial w}{\partial v}$

   in terms of the partial derivatives of w with respect to x, y, z and the partial derivatives of x, y, z with respect to u, v. In part (b) it is enough to recognize the pattern and make the appropriate modifications.

17. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Prove that $u(x, y) = f(xy)$ satisfies the partial differential equation
$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0.$$

18. Prove that if $E \subset \mathbb{R}^n$, and $f : E \to \mathbb{R}$ is differentiable at x and has a maximum or minimum at x, then $f'(x) = 0$. Hint: you know it's true when $n = 1$.

19. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be differentiable and satisfy the *Cauchy-Riemann equations*, that is,
$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}$$
on $\mathbb{R}^2$. Define $u, v : \mathbb{R}^2 \to \mathbb{R}$ by
$$u(r, \theta) = f(r\cos\theta, r\sin\theta) \quad \text{and} \quad v(r, \theta) = g(r\cos\theta, r\sin\theta).$$

Prove that u and v satisfy

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

as long as $r \neq 0$.

20. Verify that the function $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$f(x, y, z) = \left(\frac{2x}{\log(z^2 + 2)}, e^{-x^2} \sin yz \right)$$

is C^1.

21. Define $f : \mathbb{R} \to \mathbb{R}^2$ by $f(x) = (\cos x, \sin x)$. Prove that there does not exist z between 0 and 2π such that

$$f(2\pi) - f(0) = 2\pi f'(z).$$

22. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that f is differentiable on $\mathbb{R}$, but f' is discontinuous at 0.

23. Let $U = \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$, and define $q : U \to \mathbb{R}$ by

$$q(x, y) = \frac{x}{y}.$$

(a) Use the partial derivatives to help show that q is differentiable.

(b) Find a formula for $q'(a, b)(x, y)$ for $(a, b) \in U$, $(x, y) \in \mathbb{R}^2$.

(c) Use the formula you found in part (b) together with the Chain Rule to derive a "quotient rule" for

$$\left(\frac{f}{g}\right)'(a)x$$

where $f, g : E \to \mathbb{R}$, $E \subset X$, $a \in E^o$, both f and g are differentiable at a, and $0 \notin \operatorname{ran} g$.

24. Let $U \subset \mathbb{R}^n$ be open, and let $f : U \to \mathbb{R}$. Suppose every partial derivative $\dfrac{\partial f}{\partial x_i}$ is bounded on U. Prove that f is continuous on U. Hint: the Mean Value Theorem itself does *not* apply directly — instead, you should use the technique of the *proof*.

25. Let $U \subset \mathbb{R}^n$ be open and connected and $f : U \to \mathbb{R}^k$ be differentiable. Prove that if f' is identically 0 then f is constant.

Hint: Fix $a \in U$, and consider the set $V := \{x \in U : f(x) = f(a)\}$.