MAT 473 Intermediate Real Analysis II

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Linear maps — Exercises

1. Fix $x \in \mathbb{R}^n$, and define $S \in L(\mathbb{R}, \mathbb{R}^n)$ and $T \in L(\mathbb{R}^n, \mathbb{R})$ by $St = tx$ and $Ty = x \cdot y$. Prove that
$$\|S\| = \|T\| = \|x\|.$$

2. Let $T \in L(X, Y)$ be onto. Prove that if $U \subset X$ is open then so is $T(U)$. Hint: Let $W = \ker T$. Then there is a subspace Z of X such that
$$\dim Z = \dim Y, \quad X = W + Z, \quad \text{and} \quad W \cap Z = \{0\}.$$

3. (a) Let Z be a subspace of X with the same dimension as Y. Prove that the set
$$\{T \in L(X, Y) : T(Z) = Y\}$$
is open in $L(X, Y)$.

(b) Let $F \subset X$ be linearly independent and have the same number of elements as the dimension of Y. Prove that the set
$$\{T \in L(X, Y) : T(F) \text{ is linearly independent}\}$$
is open in $L(X, Y)$.

(c) Prove that the set
$$\{T \in L(X, Y) : T \text{ is onto}\}.$$
is open in $L(X, Y)$.

4. Let $m < n$, and let
$$1 \leq j_1 < j_2 < \cdots < j_m \leq n.$$
Prove that
$$\{A \in M_{m \times n} \mid A_{j_1}, \ldots, A_{j_m} \text{ are linearly independent}\}$$
is open in $M_{m \times n}$, where A_j denotes the jth column of A.

5. Let $A \in M_n$. Prove that
$$|x^t A x| \leq \|A\| \|x\|^2 \qquad \text{for all } x \in \mathbb{R}^n,$$
where x^t denotes the transpose of x.

6. A symmetric $n \times n$ matrix A is called *positive definite* if $x^t A x > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

(a) Prove that if $x^t A x > 0$ whenever $\|x\| = 1$, then A is positive definite.

(b) Formulate an appropriate definition of *negative definite*, and state and prove a result analogous to part (a) if A is negative definite. Hint: to make it easier, show that A is negative definite if and only if $-A$ is positive definite (but **this should not be your definition of negative definite!**).

7. Let $A \in M_n$.

   (a) Suppose A is positive definite. Prove that there exists $\varepsilon > 0$ such that for every symmetric $B \in M_n$, if $\|B - A\| < \varepsilon$ then B is positive definite.

   Hint: show that $\inf\{x^t Ax : \|x\| = 1\} > 0$.

   (b) Formulate and prove a result analogous to part (a) if A is negative definite.

8. Let $A \in M_n$ and $x \in \mathbb{R}^n$.

   (a) Suppose $x^t Ax > 0$. Prove that there exists $\varepsilon > 0$ such that for all $B \in M_n$, if $\|B-A\| < \varepsilon$ then $x^t Bx > 0$.

   (b) State and prove a result analogous to part (a) if $x^t Ax < 0$.