TOPOLOGICAL GRAPHS

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Abstract. We give a rough introduction to topological graphs and their C^*-algebras.

1. Introduction

I'll mostly follow the conventions of [1] (see also [2]) for topological graphs, with some exceptions influenced by personal taste and the informal nature of these notes.

2. Homomorphisms between commutative C^*-algebras

We begin with some generalities on homomorphisms between commutative C^*-algebras. Let X and Y be locally compact Hausdorff spaces, and let $r : X \to Y$ be continuous. Define $\phi : C_0(Y) \to C_b(X)$ by the commuting diagram

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[dr, "\phi(f)"'] & Y \arrow[d, "f"] \\
& \mathbb{C}.
\end{tikzcd}$$

Questions:

1. When does ϕ map into $C_0(X)$?
2. When is ϕ injective?

Better questions: what open subsets of Y correspond to the ideals

3. $\phi^{-1}(C_0(X))$?
4. $\ker \phi$?

Answers:

Lemma 2.1.

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3. $Y_{\text{fin}} := \{x \in Y : r^{-1}(U)$ *is compact for some neighborhood* U *of* $x\}$
4. $Y_{\text{sce}} := Y \setminus \overline{r(X)}$

Proof. 3. Let $f \in C_0(Y)$. We must show that $f \circ r \in C_0(X)$ if and only if $f \equiv 0$ on $Y \setminus Y_{\text{fin}}$. For the forward direction, we argue contrapositively: suppose $x \in Y \setminus Y_{\text{fin}}$ and $f(x) \neq 0$. Put $a = |f(x)|/2$ and

$$U = \{|f| \geq a\}.$$

Then U is a compact neighborhood of x. Since $x \notin Y_{\text{fin}}$, $r^{-1}(U)$ is not compact. But

$$r^{-1}(U) = \{|f \circ r| \geq a\},$$

so $f \circ r \notin C_0(X)$.

Conversely, suppose $f \equiv 0$ on $Y \setminus Y_{\text{fin}}$, let $a > 0$, and put $U = \{|f| \geq a\}$. Then U is a compact subset of the open set Y_{fin}. Each $x \in U$ has a compact neighborhood V_x such that $r^{-1}(V_x)$ is compact. Since U is compact, there are $x_1, \dots, x_n \in U$ such that

$$U \subset \bigcup_1^n V_{x_i},$$

so that

$$r^{-1}(U) \subset \bigcup_1^n r^{-1}(V_{x_i}).$$

Thus the closed subset $r^{-1}(U)$ is compact since $\bigcup_1^n r^{-1}(V_{x_i})$ is. But again

$$r^{-1}(U) = \{|f \circ r| \geq a\}.$$

Thus $f \circ r \in C_0(X)$.

4. This is easy: for $f \in C_0(Y)$ we have $\phi(f) = 0$ if and only if $f \equiv 0$ on $r(X)$, equivalently $f \equiv 0$ on $\overline{r(X)}$ since f is continuous. □

Now put

$$Y_{\text{rg}} = Y_{\text{fin}} \setminus \overline{Y_{\text{sce}}}.$$

Corollary 2.2. *$C_0(Y_{\text{rg}})$ is the largest ideal of $C_0(Y)$ that ϕ maps injectively into $C_0(X)$.*

3. TOPOLOGICAL GRAPHS AND THEIR CORRESPONDENCES

Definition 3.1. A *topological graph* comprises locally compact Hausdorff spaces E, Y and maps $r, s : E \to Y$ such that r is continuous and s is a local homeomorphism.

We usually just refer to E as the topological graph, with the understanding that the other ingredients Y, r, s are around. The elements of E and Y are called *edges* and *vertices*, respectively.

Warning: our notation differs from the literature, in particular from [1].

It will be convenient to write

$$\begin{aligned} X_0 &= C_c(E) \\ A &= C_0(Y). \end{aligned}$$

We make X_0 into a pre-A-correspondence[1] with the following operations:

$$\begin{aligned} f \cdot \xi \cdot g &= (f \circ r)\xi(g \circ s) \\ \langle \xi, \eta \rangle(v) &= \sum_{s(e)=v} \overline{\xi(e)}\eta(e). \end{aligned}$$

Note that the above sum is actually finite since $s^{-1}(v) \cap K$ is finite for every compact $K \subset E$. Also note that $\langle \cdot, \cdot \rangle$ is sesquilinear (linear in the 2nd variable). It is not quite obvious that the inner product actually takes values in A:

Lemma 3.2. *If* $\xi, \eta \in X_0$ *then* $\langle \xi, \eta \rangle \in A$.

Proof. By the polarization identity

$$\langle \xi, \eta \rangle = \frac{1}{4} \sum_0^3 i^k \langle \xi + i^k \eta, \xi + i^k \eta \rangle$$

it suffices to show $\langle \xi, \xi \rangle \in A$. Let $v \in Y$, and put

$$K = \operatorname{supp} \xi \cap s^{-1}(v).$$

Then K is finite, so we can choose, for each $e \in K$, a neighborhood U_e of e on which s is a homeomorphism, such that the U_e's are disjoint

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[1] that is, a pre-Hilbert A-module equipped with a left module action of A by bounded adjointable operators—see the appendix

and every $s(U_e)$ is the same neighborhood W. Then

$$\langle \xi, \xi \rangle|_W = \sum_{e \in K} \left| \xi \circ (s|_{U_e})^{-1} \right|^2,$$

which is continuous. Since $\operatorname{supp}\langle \xi, \xi \rangle \subset s(\operatorname{supp} \xi)$, we have $\langle \xi, \xi \rangle \in C_c(Y) \subset A$. ☐

A clearly acts on the left of X_0 by adjointable operators, and it is easy to see that the left module action is bounded; in fact, $C_b(E)$ acts by bounded adjointable operators via pointwise multiplication, because if $f \in C_b(E)$ and $\xi \in X_0$ then

$$\begin{aligned} \langle f\xi, f\xi \rangle(v) &= \sum_{s(e)=v} |f(e)|^2 |\xi(e)|^2 \\ &\le \|f\|_u^2 \langle \xi, \xi \rangle(v). \end{aligned}$$

Definition 3.3. The *graph correspondence* is the completion $X = X_E$ of the pre-A-correspondence X_0.

Note that the A-correspondence X is nondegenerate in the sense that $A \cdot X = X$, because if $\xi \in C_c(E)$ we can take $f \in X_0$ such that $f \equiv 1$ on $r(\operatorname{supp} \xi)$, and then $f \cdot \xi = \xi$. Also, the action of $C_b(E)$ by adjointable operators on X_0 extends to an action on X, so we have a nondegenerate homomorphism

$$M : C_b(E) \to \mathcal{L}(X)$$

determined by $M_f \xi = f\xi$ for $\xi \in X_0$.

Lemma 3.4. *For $f \in C_b(E)$ we have $M_f \in \mathcal{K}(X)$ if and only if $f \in C_0(E)$.*

Proof. Since $M : C_b(E) \to \mathcal{L}(X)$ is a homomorphism of C^*-algebras, to show that if $f \in C_0(E)$ then $M_f \in \mathcal{K}(X)$ it suffices to consider $f \in C_c(E)$. The support of f can be covered by finitely many open sets $U_1, \dots, U_n$ on which s is a homeomorphism. Let $\{\phi_i\}$ be a partition of unity on $\operatorname{supp} f$ subordinate to the open cover $\{U_i\}$. For each i define

$$\xi_i = f\phi_i^{1/2} \quad \text{and} \quad \eta_i = \phi_i^{1/2},$$

so that

$$f = \sum_1^n \xi_i \eta_i.$$

Note that, for every $i = 1, \dots, n$ and e, e' with $s(e) = s(e')$ we have

$$\xi_i(e)\eta_i(e') = 0 \qquad \text{unless } e = e'.$$

We will show that $M_f = \sum_1^n \theta_{\xi_i,\eta_i}$, and it suffices to show that $M_{\xi_i\eta_i} = \theta_{\xi_i,\eta_i}$ for each i: if $\zeta \in X_0$ and $e \in E$ then

$$\begin{aligned}\theta_{\xi_i,\eta_i}\zeta(e) &= \bigl(\xi_i\langle\eta_i,\zeta\rangle\bigr)(e)\\ &= \xi_i(e)\sum_{s(e')=s(e)} \overline{\eta_i(e')}\zeta(e')\\ &= \xi_i(e)\eta_i(e)\zeta(e)\\ &= \bigl(M_{\xi_i\eta_i}\zeta\bigr)(e).\end{aligned}$$

Conversely, let $f \notin C_0(E)$, and choose $\varepsilon > 0$ such that the set

$$C = \{\, |f| \ge \varepsilon\}$$

is not compact. To show that $M_f \notin \mathcal{K}(X)$, since X_0 is dense in X it suffices to show that for all $\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_n \in X_0$ we have

$$\left\| M_f - \sum_1^n \theta_{\xi_i,\eta_i} \right\| \ge \varepsilon.$$

Choose

$$e_0 \in C \Bigm\backslash \bigcup_1^n \operatorname{supp}\eta_i,$$

then choose an open neighborhood U of e_0 on which r is a homeomorphism, and which is disjoint from $\bigcup_1^n \operatorname{supp}\eta_i$, and finally choose $\zeta \in C_c(U)$ with $0 \le \zeta \le 1$ and $\zeta(e_0) = 1$. Then

$$\|\zeta\| = \|\zeta\|_u = 1$$

because s is injective on $\operatorname{supp}\zeta$, and, because

$$\sum_1^n \theta_{\xi_i,\eta_i}\zeta = 0,$$

we have

$$\left\| \left(M_f - \sum_1^n \theta_{\xi_i,\eta_i} \right) \zeta \right\| = \|M_f\zeta\| \ge |f(e_0)\zeta(e_0)| \ge \varepsilon. \qquad \square$$

Now let $\phi : A \to \mathcal{L}(X)$ be the left module action. Recall from Section 2 the notation

- $Y_{\text{fin}} = \{v \in Y : \overline{r^{-1}(U)} \text{ is compact for some neighborhood } U \text{ of } v\}$;
- $Y_{\text{sce}} = Y \setminus \overline{r(E)}$;
- $Y_{\text{rg}} = Y_{\text{fin}} \setminus \overline{Y_{\text{sce}}}$.

Then by Lemma 2.1 we have:

Theorem 3.5. *$C_0(Y_{\mathrm{rg}})$ is the largest ideal of A which ϕ maps injectively into $\mathcal{K}(X)$.*

4. TOPOLOGICAL GRAPH C^*-ALGEBRAS

Definition 4.1. A *Toeplitz representation* of E in a C^*-algebra B comprises a linear map $\psi : X \to B$ and a homomorphism $\pi : A \to B$ such that for $f, g \in A$ and $\xi, \eta \in X$ we have:

1. $\psi(f \cdot \xi) = \pi(f)\psi(\xi)$;
2. $\psi(\xi)^*\psi(\eta) = \pi(\langle \xi, \eta \rangle)$;
3. $\psi(\xi \cdot g) = \psi(\xi)\pi(g)$.

Actually, 3 follows from 2,[2] but is included for emphasis.

A Toeplitz representation (ψ, π) of X in B gives rise to a homomorphism $\Psi : \mathcal{K}(X) \to B$ by

$$\Psi(\theta_{\xi,\eta}) = \psi(\xi)\psi(\eta)^*.$$

Definition 4.2. A *Cuntz-Pimsner representation*, or *C-P rep* for short, of X is a Toeplitz rep (σ, π) such that

$$\pi(f) = \Psi \circ \phi(f) \qquad \text{for all } f \in C_0(Y_{\mathrm{rg}}).$$

The *C^*-algebra* of X is a C^*-algebra $\mathcal{O}_X$ equipped with a C-P rep (i_X, i_A) having the universal property that for every C-P rep (ψ, π) in a C^*-algebra B there is a unique homomorphism $\psi \times \pi$ making the diagram

$$\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[dr, "\psi"'] & \mathcal{O}_X \arrow[d, dashed, "!\ \psi\times\pi"] & A \arrow[l, "i_A"'] \arrow[dl, "\pi"] \\
& B &
\end{tikzcd}$$

commute.

Thus the C^*-algebra of X encodes the representation theory, as it should.

When E (hence also Y) is discrete, we just have a directed graph[3], and the C-P representations correspond exactly to the usual Cuntz-Krieger E-families, because a vertex v is in Y_{rg} if and only if

$$0 < \left|r^{-1}(v)\right| < \infty,$$

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[2] exercise

[3] see the appendix

and for $v \in Y_{\mathrm{rg}}$ we have, using δ-notation for delta functions, $\pi(\delta_v) = \Psi \circ \phi(\delta_v)$ if and only if

$$\pi(\delta_v) = \sum_{r(e)=v} \psi(\delta_e)\psi(\delta_e)^*.$$

Thus $\mathcal{O}_X$ is the usual graph algebra $C^*(E)$.

5. Appendix

Here we give the basic definitions of Hilbert modules, correspondences, and graph algebras. Raeburn's conference lectures [2] is highly recommended for more detail.

5.1. **Hilbert modules and correspondences.**

Let A be a C^*-algebra. A *pre-Hilbert A-module* is a right A-module X equipped with an *A-valued inner product*, i.e., a sesquilinear map $\langle \cdot, \cdot \rangle : X \times X \to A$ such that for $\xi, \eta \in X$ and $a \in A$ we have:

1. $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$;
2. $\langle \xi, \xi \rangle \geq 0$, and $\langle \xi, \xi \rangle = 0$ implies $\xi = 0$;
3. $\langle \xi, \eta \cdot a \rangle = \langle \xi, \eta \rangle a$.

It turns out that a version of the Cauchy-Schwarz inequality holds:

$$\|\langle \xi, \eta \rangle\|^2 \leq \|\langle \xi, \xi \rangle\| \|\langle \eta, \eta \rangle\|.$$

Consequently, X becomes a normed space with

$$\|\xi\| = \langle \xi, \xi \rangle^{1/2}.$$

X is a *Hilbert A-module* if it is complete in this norm. In any event, the completion becomes a Hilbert A-module with the natural extensions of the operations.[4]

An A-module operator T on a pre-Hilbert A-module X is *adjointable* if there is an A-module operator S on X such that

$$\langle T\xi, \eta \rangle = \langle \xi, S\eta \rangle \qquad \text{for all } \xi, \eta \in X.$$

If an adjointable operator on a pre-Hilbert module extends to an adjointable operator on the Hilbert-module completion. An adjointable operator on a Hilbert module is automatically bounded[5]. The set $\mathcal{L}(X)$

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[4] There are various weakenings of the axioms that are occasionally useful, although we don't need them here. For example, sometimes A has to be initially allowed to be a dense *-subalgebra of a C^*-algebra, then the completed Hilbert module uses the completed C^*-algebra.

[5] by the Uniform Boundedness principle

of all adjointable operators on an A-correspondence is a C^*-algebra with the operator norm.

If X is a Hilbert A-module and $\xi, \eta \in X$, the map $\theta_{\xi,\eta} : X \to X$ defined by

$$\theta_{\xi,\eta}(\zeta) = \xi\langle \eta, \zeta\rangle$$

is adjointable, with adjoint $\theta_{\eta,\xi}$. The closed span $\mathcal{K}(X)$ of all these maps is an ideal of $\mathcal{L}(X)$, and an element of $\mathcal{K}(X)$ is called a *compact operator*.[6]

A *pre-A-correspondence* is a pre-Hilbert A-module X which is also equipped with a left A-module action by adjointable operators, so that

$$\langle a \cdot \xi, \eta\rangle = \langle \xi, a^* \cdot \eta\rangle,$$

and X is called simply an *A-correspondence* if it is complete, i.e., is actually a Hilbert A-module. In any event, if the left A-module action on a pre-A-correspondence X is bounded, then the completion of X becomes an A-correspondence.

An A-correspondence X is *nondegenerate* if

$$\overline{\operatorname{span}}\{a \cdot \xi : a \in A, \xi \in X\} = X.$$

Similarly for a pre-A-correspondence. It turns out that, by the Cohen-Hewitt factorization theorem, if an A-correspondence X is nondegenerate then actually every vector $\xi \in X$ can be factored:

$$\xi = a \cdot \eta \qquad \text{for some } a \in A, \eta \in X,$$

and so we can just write $A \cdot X = X$ for nondegeneracy.[7]

We could go on to discuss representations and C^*-alebras of arbitrary correspondences, but we eschew it; the discussion in Section 3 for graph correspondences is all we need.

5.2. **Graph algebras.** A *directed graph* comprises sets E and Y and maps $r, s : E \to Y$.[8]

A *Cuntz-Krieger E-family*, in a C^*-algebra B comprises partial isometries $\{t_e\}_{e\in E}$ and mutually orthogonal projections $\{q_v\}_{v\in Y}$ such that:

1. $t_e^* t_e = q_{s(e)}$ for all $e \in E$;
2. $q_v = \sum_{r(e)=v} t_e t_e^*$ for all $v \in Y$ with $0 < |r^{-1}(v)| < \infty$.

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[6] But it is *not* compact in the usual sense of linear operators on Banach spaces!

[7] But the Cohen-Hewitt theorem requires completeness of X (and A)!

[8] Usually E and Y are required to be countable. Note that a directed graph may be regarded as a discrete topological graph.

The C^**-algebra* of E is a C^*-algebra $C^*(E)$ equipped with a Cuntz-Krieger E-family $\{s_e, p_v\}$ having the universal property that for every Cuntz-Krieger E family $\{t_e, q_v\}$ in a C^*-algebra B there is a unique homomorphism $\pi : C^*(E) \to B$ such that

$$\begin{aligned} \pi(s_e) &= t_e \qquad \text{for all } e \in E; \\ \pi(p_v) &= q_v \qquad \text{for all } v \in Y. \end{aligned}$$

REFERENCES

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