

# BRUTAL INTRODUCTION TO NONCOMMUTATIVE DUALITY

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ABSTRACT. Noncommutative duality for  $C^*$ -dynamical systems is a vast generalization of Pontryagin duality for locally compact abelian groups. We aim to show how paying attention to the categorical perspective gives added insight into duality.

These notes are very rough, and make no claim to completeness, or even correctness. And no references are given, although all the facts described below are in the literature.

Throughout,  $G$  will always be a locally compact group, and  $A, B$  will be  $C^*$ -algebras.

The *basic  $C^*$ -category*  $\mathbf{C}^*$  comprises  $C^*$ -algebras and nondegenerate homomorphisms into multiplier algebras, i.e.,  $\pi : A \rightarrow B$  in  $\mathbf{C}^*$  means  $\pi : A \rightarrow M(B)$  and  $\pi(A)B = B$ .

**Actions and crossed products.** An action  $(A, \alpha)$  of  $G$  on  $A$  is a homomorphism  $\alpha : G \rightarrow \text{Aut } A$ . The equivariant category  $\mathbf{Act}$  comprises actions and equivariant morphisms from  $\mathbf{C}^*$ , i.e.,  $\pi : (A, \alpha) \rightarrow (B, \beta)$  in  $\mathbf{Act}$  means  $\pi : A \rightarrow B$  in  $\mathbf{C}^*$  and

$$\pi \circ \alpha_s = \beta_s \circ \pi.$$

*Inner actions* arise via conjugating by unitary homomorphisms  $u : G \rightarrow M(A)$ :

$$\text{Ad } u_s(a) = u_s a u_s^*.$$

By universal properties,  $u : G \rightarrow M(A)$  corresponds to a morphism

$$u : C^*(G) \rightarrow A$$

in the category  $\mathbf{C}^*$ , and hence to an object in the comma category

$$C^*(G) \downarrow \mathbf{C}^*.$$

$(A, u) \mapsto (A, \text{Ad } u)$  extends uniquely to the *inner action functor*

$$\text{Ad} : (C^*(G) \downarrow \mathbf{C}^*) \rightarrow \mathbf{Act}.$$

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A *covariant homomorphism*  $(\pi, u)$  of an action  $(A, \alpha)$  in  $B$  comprises morphisms  $u : C^*(G) \rightarrow B$  in  $\mathbf{C}^*$  and  $\pi : (A, \alpha) \rightarrow (B, \text{Ad } u)$  in  $\mathbf{Act}$ .

The *crossed-product functor*

$$\text{CP} : \mathbf{Act} \rightarrow (C^*(G) \downarrow \mathbf{C}^*)$$

is a left adjoint to  $\text{Ad}$ . The universal property is expressed by the commutative diagram

$$\begin{array}{ccc} (A, \alpha) & \xrightarrow{i_A} & (A \times_\alpha G, \text{Ad } i_G) & & (A \times_\alpha G, i_G) \\ & \searrow \pi & \downarrow \pi \times u & & \downarrow \pi \times u \\ & & (B, \text{Ad } u) & & (B, u), \end{array}$$

so that

$$\text{CP}(A, \alpha) = (A \times_\alpha G, i_G),$$

and

$$i_A : \text{id}_{\mathbf{Act}} \rightarrow \text{Ad} \circ \text{CP}$$

is the unit of the adjunction. Universality is more commonly expressed by the diagram

$$\begin{array}{ccccc} A & \xrightarrow{i_A} & A \times_\alpha G & \xleftarrow{i_G} & C^*(G) \\ & \searrow \pi & \downarrow \pi \times u & & \swarrow u \\ & & B & & \end{array}$$

in  $\mathbf{C}^*$ .

For functoriality, we need a map on morphisms:

$$\begin{array}{ccc} (A, \alpha) & & (A \times_\alpha G, i_G) \\ \phi \downarrow & & \downarrow \phi \times G = (i_B \circ \phi) \times i_G \\ (B, \beta) & & (B \times_\beta G, i_G). \end{array}$$

**Coactions and their crossed products.** If  $G$  is abelian, we have correspondences

$$\begin{array}{ccc}
 C^*(G) & \xleftarrow{\text{Fourier transform}} & C_0(\widehat{G}) \\
 \\ 
 C_0(G) & \xleftarrow{\hspace{10em}} & C^*(\widehat{G}) \\
 \\ 
 \text{coaction of } G & \xleftarrow{\hspace{10em}} & \text{action of } \widehat{G} \\
 \\ 
 \text{action of } G & \xleftarrow{\hspace{10em}} & \text{coaction of } \widehat{G}
 \end{array}$$

If  $G$  is nonabelian, coactions of  $G$  correspond roughly to actions of the nonexistent dual group. The theory of coactions is satisfyingly parallel to actions:

There is an equivariant category **Coact** comprising coactions  $(A, \delta)$  and equivariant morphisms from  $\mathbf{C}^*$ .

Morphisms  $\mu : C_0(G) \rightarrow A$  in  $\mathbf{C}^*$  give *inner coactions*  $(A, \text{Ad } \mu)$ , we have an *inner coaction functor*

$$\text{Ad} : (C_0(G) \downarrow \mathbf{C}^*) \rightarrow \mathbf{Coact},$$

and the *crossed-product functor*

$$\text{CP} : \mathbf{Coact} \rightarrow (C_0(G) \downarrow \mathbf{C}^*)$$

is a left adjoint, with universal property

$$\begin{array}{ccc}
 (A, \delta) & \xrightarrow{j_A} & (A \times_\delta G, \text{Ad } j_G) & (A \times_\delta G, j_G) \\
 & \searrow \pi & \downarrow \pi \times \mu & \downarrow \pi \times \mu \\
 & & (B, \text{Ad } \mu) & (B, \mu),
 \end{array}$$

so that

$$\text{CP}(A, \delta) = (A \times_\delta G, j_G),$$

and the unit of the adjunction is

$$j_A : \text{id}_{\mathbf{Coact}} \rightarrow \text{Ad} \circ \text{CP}.$$

Functoriality is expressed by

$$\begin{array}{ccc} (A, \delta) & & (A \times_{\delta} G, j_G) \\ \downarrow \phi & & \downarrow \phi \times G = (j_B \circ \phi) \times i_G \\ (B, \varepsilon) & & (B \times_{\varepsilon} G, j_G). \end{array}$$

**Duality.** Given an action  $(A, \alpha)$ , the *dual coaction*

$$(A \times_{\alpha} G, \widehat{\alpha})$$

is characterized by:

- $\widehat{\alpha}$  is trivial on  $i_A(A)$ ;
- $i_G : (C^*(G), \delta_G) \rightarrow (A \times_{\alpha} G, \widehat{\alpha})$  in **Coact**, where  $\delta_G$  is the *comultiplication* on  $C^*(G)$ :

$$\delta_G(s) = s \otimes s \quad \text{for } s \in G.$$

Now we regard CP as a functor

$$\begin{aligned} (A, \alpha) &\mapsto (A \times_{\alpha} G, \widehat{\alpha}, i_G) \\ \mathbf{Act} &\xrightarrow{\text{CP}} (C^*(G), \delta_G) \downarrow \mathbf{Coact}. \end{aligned}$$

*Crossed-product duality* is a *Morita equivalence*

$$A \times_{\alpha} G \times_{\widehat{\alpha}} G \sim A.$$

What about a dual version, starting with a coaction  $(A, \delta)$ ?

There is a *dual action*

$$(A \times_{\delta} G, \widehat{\delta})$$

characterized by:

- $\widehat{\delta}$  is trivial on  $j_A(A)$ ;
- $j_G : (C_0(G), \text{rt}) \rightarrow (A \times_{\delta} G, \widehat{\delta})$  in **Act**, where *rt* is *right translation* on  $C_0(G)$ :

$$\text{rt}_s(f)(t) = f(ts),$$

and we get a *crossed-product functor*

$$\begin{aligned} (A, \delta) &\mapsto (A \times_{\delta} G, \widehat{\delta}, j_G) \\ \mathbf{Coact} &\xrightarrow{\text{CP}} (C_0(G), \text{rt}) \downarrow \mathbf{Act}. \end{aligned}$$

But now we need a diversion: the *reduced crossed product* of an action  $(A, \alpha)$  is the image of the *regular representation*

$$A \times_{\alpha} G \xrightarrow{\Lambda} A \times_{\alpha, r} G,$$

generalizing the regular representation of  $G$ :

$$C^*(G) \xrightarrow{\lambda} C_r^*(G) \subset B(L^2(G)),$$

and is onto, but *not 1-1* in general. The dual coaction  $\widehat{\alpha}$  descends to a coaction  $\widehat{\alpha}^n$  (the reason for the superscript “ $n$ ” is that this is the “normalization” of  $\widehat{\alpha}$  — see later) on the reduced crossed product, and

$$\Lambda : (A \times_{\alpha} G, \widehat{\alpha}) \rightarrow (A \times_{\alpha,r} G, \widehat{\alpha}^n)$$

in **Coact**. Composing with  $\Lambda$  gives a covariant homomorphism  $(i_A^r, i_G^r)$  of  $(A, \alpha)$  in  $A \times_{\alpha,r} G$ . We get a *reduced-crossed-product functor*

$$(A, \alpha) \mapsto (A \times_{\alpha,r} G, \widehat{\alpha}^n, i_G^r)$$

$$\mathbf{Act} \xrightarrow{\text{RCP}} (C^*(G), \delta_G) \downarrow \mathbf{Coact}.$$

A coaction  $(A, \delta)$  can come in various types, and here are the opposite extremes:

- *maximal*, satisfying *full-crossed-product duality*:

$$A \times_{\delta} G \times_{\widehat{\delta}} G \sim A;$$

- *normal*, satisfying *reduced-crossed-product duality*:

$$A \times_{\delta} G \times_{\widehat{\delta},r} G \sim A.$$

Maximal and normal coactions form full subcategories **MCoact** and **NCoact** of **Coact**.

Just like a set can be neither open nor closed, a coaction can be neither maximal nor normal!

Every coaction  $(A, \delta)$  has both a

- *maximalization*:

$$(A^m, \delta^m) \xrightarrow[\text{onto}]{q^m} (A, \delta)$$

such that

$$q^m \times G : A^m \times_{\delta^m} G \xrightarrow{\cong} A \times_{\delta} G,$$

and a

- *normalization*:

$$(A, \delta) \xrightarrow[\text{onto}]{q^n} (A^n, \delta^n)$$

such that

$$q^n \times G : A \times_{\delta} G \xrightarrow{\cong} A^n \times_{\delta^n} G.$$

Maximalization has the universal property

$$\begin{array}{ccc} (B, \varepsilon) & & \\ \downarrow & \searrow & \\ \Downarrow & & \\ (A^m, \delta^m) & \xrightarrow{q^m} & (A, \delta). \end{array}$$

Thus  $(A, \delta) \mapsto (A^m, \delta^m)$  extends uniquely to a functor

$$\text{Max} : \mathbf{Coact} \rightarrow \mathbf{MCoact}$$

such that

$$q^m : \text{Inc}_{\mathbf{MCoact}} \circ \text{Max} \rightarrow \text{id}_{\mathbf{Coact}}$$

is a natural transformation,  $\mathbf{MCoact}$  is a coreflective subcategory of  $\mathbf{Coact}$ , and Max is a coreflector.

Dually, normalization has the universal property

$$\begin{array}{ccc} (A, \delta) & \xrightarrow{q^n} & (A^n, \delta^n) \\ & \searrow & \downarrow \\ & & \Downarrow \\ & & (B, \varepsilon), \end{array}$$

$(A, \delta) \mapsto (A^n, \delta^n)$  extends uniquely to a functor

$$\text{Nor} : \mathbf{Coact} \rightarrow \mathbf{NCoact}$$

such that

$$q^n : \text{id}_{\mathbf{Coact}} \rightarrow \text{Inc}_{\mathbf{NCoact}} \circ \text{Nor}$$

is natural,  $\mathbf{NCoact}$  is a reflective subcategory of  $\mathbf{Coact}$ , and Nor is a reflector.

Moreover, Max and Nor satisfy the following two extra properties:

- if  $(A, \delta)$  is maximal, then  $q^n : (A, \delta) \rightarrow (A^n, \delta^n)$  is not only a normalization of  $(A, \delta)$ , but is also a maximalization of  $(A^n, \delta^n)$ ;
- if  $(A, \delta)$  is normal, then  $q^m : (A^m, \delta^m) \rightarrow (A, \delta)$  is not only a maximalization of  $(A, \delta)$ , but is also a normalization of  $(A^m, \delta^m)$ ;

consequently:

**Theorem** (Maximal-Normal Equivalence). *Normalization restricts to an equivalence*

$$\mathbf{MCoact} \sim \mathbf{NCoact},$$

*and maximalization restricts on  $\mathbf{NCoact}$  to a quasi-inverse.*

**Landstad duality.** Crossed-product duality allows recovery of the original  $C^*$ -dynamical system up to Morita equivalence, just knowing the dual  $C^*$ -dynamical system. To recover the system up to isomorphism requires keeping track of one more piece of information.

**Theorem** (For actions). *Both the crossed-product functor*

$$(A, \alpha) \mapsto (A \times_\alpha G, \widehat{\alpha}, i_G)$$

$$\mathbf{Act} \xrightarrow{\text{CP}} (C^*(G), \delta_G) \downarrow \mathbf{MCoact}$$

and the reduced-crossed-product functor

$$(A, \alpha) \mapsto (A \times_{\alpha,r} G, \widehat{\alpha}^r, i_G^r)$$

$$\mathbf{Act} \xrightarrow{\text{RCP}} (C^*(G), \delta_G) \downarrow \mathbf{NCoact}$$

are equivalences. Moreover, the diagram

$$\begin{array}{ccc} \mathbf{Act} & \xrightarrow{\text{CP}} & (C^*(G), \delta_G) \downarrow \mathbf{MCoact} \\ & \searrow \text{RCP} & \downarrow \text{Nor} \\ & & (C^*(G), \delta_G) \downarrow \mathbf{NCoact} \end{array}$$

commutes.

**Theorem** (For coactions). *The crossed-product functor*

$$(A, \delta) \mapsto (A \times_\delta G, \widehat{\delta}, j_G)$$

$$\mathbf{NCoact} \xrightarrow{\text{CP}} (C_0(G), \text{rt}) \downarrow \mathbf{Act}$$

is an equivalence.

It is interesting to interpret the familiar properties of category equivalences in the case of Landstad duality. For actions, we have a Landstad duality for both full and reduced crossed products. Let's do our interpreting for full crossed products: being an equivalence,  $\text{CP} : \mathbf{Act} \rightarrow (C^*(G), \delta_G) \downarrow \mathbf{MCoact}$  is full, faithful, and essentially surjective, i.e.,

- for any two actions  $(A, \alpha)$  and  $(B, \beta)$ , we have a bijection

$$\begin{array}{ccc} (A, \alpha) & & (A \times_\alpha G, \widehat{\alpha}, i_G) \\ \downarrow \phi & \xleftarrow{\cong} & \downarrow \phi \times G \\ (B, \beta) & & (B \times_\beta G, \widehat{\beta}, i_G) \end{array}$$

between morphisms, and

- every triple  $(B, \delta, u)$  in  $(C^*(G), \delta_G) \downarrow \mathbf{MCoact}$  is isomorphic to one of the form  $(A \times_\alpha G, \widehat{\alpha}, i_G)$ .

From the properties of comma categories we can deduce a characterization of crossed products: a  $C^*$ -algebra  $B$  is isomorphic to a full crossed product  $A \times_\alpha G$  by an action  $(A, \alpha)$  if and only if there exist:

- a maximal coaction  $\delta$  of  $G$  on  $B$ , and
- a strictly continuous unitary homomorphism  $u : G \rightarrow M(B)$  such that  $\delta(u_s) = u_s \otimes s$  for all  $s \in G$ .

From Landstad duality for reduced crossed products we get a similar characterization: a  $C^*$ -algebra  $B$  is isomorphic to a reduced crossed product  $A \times_{\alpha,r} G$  by an action  $(A, \alpha)$  if and only if there exist:

- a normal coaction  $\delta$  of  $G$  on  $B$ , and
- a strictly continuous unitary homomorphism  $u : G \rightarrow M(B)$  such that  $\delta(u_s) = u_s \otimes s$  for all  $s \in G$ .

This is actually very close to Landstad's original theorem, which spawned all of the above work on "Landstad duality".

And here is the dual version, a characterization of coaction-crossed products: a  $C^*$ -algebra  $B$  is isomorphic to a crossed product  $A \times_\delta G$  by a coaction  $(A, \delta)$  if and only if there exist:

- an action  $\alpha$  of  $G$  on  $B$ , and
- a nondegenerate homomorphism  $\mu : C_0(G) \rightarrow M(B)$  that is  $\text{rt} - \alpha$  equivariant.