CATEGORICAL PERSPECTIVES IN NONCOMMUTATIVE DUALITY

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ABSTRACT. In the study of C*-dynamical systems, especially in crossed-product duality, everyone knows that category theory can afford an efficiency and unity of language, although some would characterize the effect as obfuscatory. It's perhaps not quite so widely known that categorical techniques can actually improve some of the results. Similarly, everyone knows that the crossed-product construction is a functor. But a more serious use of categorical techniques reveals that there are category equivalences around, and even adjoint functors.

1. Efficiency of language

Here's an example of how category theory can afford an efficiency of language: first we'll state a theorem, then we'll give a much simpler (well, certainly shorter, but perhaps "simpler" is a matter of opinion...) statement in which we take advantage of categorical language.

The theorem, originally proven by Landstad in [9, Theorem 3], has become one of the standard results in C^*-dynamical systems, especially regarding "noncommutative duality". It gives a characterization of reduced crossed products. He refers to it as a "duality theorem for C^*-crossed products", and I subsequently named the result "Landstad duality". The version we state here appeared in [5] as a stepping stone to Landstad duality for *full* crossed products; it's a significant restructuring of Landstad's version, and we needed to work a bit to prove it.

Theorem 1.1 ([5, Theorem 3.1]). *Let B be a C^*-algebra and G a locally compact group. Then*

(i) *B is isomorphic to a reduced crossed product $A \times_{\alpha,r} G$ by an action α of G on a C^*-algebra A if and only if there are a strictly continuous unitary homomorphism $u : G \to M(B)$ and*

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a normal coaction δ of G on B such that

$$\delta(u_s) = u_s \otimes s. \tag{1.1}$$

(ii) *Moreover, if the above conditions hold then an isomorphism $\theta : B \to A \times_{\alpha,r} G$ can be chosen to be $\delta - \widehat{\alpha}$ equivariant and to satisfy $\theta \circ u = i_G^r$, and if (C, G, β) is another action and $\sigma : B \to C \times_{\beta,r} G$ is a $\delta - \widehat{\beta}$ equivariant isomorphism such that $\sigma \circ u = i_G^{\beta,r}$, then there is an isomorphism $\phi : (A, \alpha) \to (C, \beta)$ such that $(\phi \times_r G) \circ \theta = \sigma$.*

Ok, that was a mouthful; the uniqueness clause (ii) is particularly cumbersome. Now we'll state a stronger, but much shorter, version of Landstad duality (Theorem 1.2 below). Here's the idea: regard the crossed product as a functor (and we'll explain how this works in a little while). Landstad duality tells us when something is isomorphic to an image of the crossed-product functor, and moreover what sort of uniqueness to expect. Here we'll state the categorical version of Landstad duality, after which we'll explain what it means in some detail. Steve and I proved the following theorem quite recently (it appeared in 2009), although here the notation is changed a bit from [6].

Theorem 1.2 ([6, Theorem 4.1])**.** *The functor*

$$(A, \alpha) \mapsto (A \times_{\alpha,r} G, \widehat{\alpha}^n, i_G^r)$$
$$\phi \mapsto \phi \times_r G$$

is an equivalence from the category $\mathbf{C^*act}$ *to the comma category* $(C^*(G), \delta_G) \downarrow \mathbf{C^*coact^n}$.

Let's figure out what all that means. The basic crossed-product functor (which we'll soon embellish) takes an action $\alpha : G \to \operatorname{Aut} A$ and produces the crossed-product C^*-algebra $A \times_\alpha G$. For a thorough exposition of the theory of actions, coactions, and their crossed products, I recommend [3, Appendix A]. Here we'll just give a "hand-wavy" introduction. There are many constructions of $A \times_\alpha G$, but the main thing is that it has a universal property:[1] it has the same representation theory as the covariant representations of the action. A *covariant representation* of an action (A, α) is a pair (π, u), where π and u are representations of A and G, respectively, on a Hilbert space, such that $\pi \circ \alpha_s = \operatorname{Ad} u_s \circ \pi$ for $s \in G$. More generally, instead of representations on Hilbert space, it's customary to use *covariant homomorphisms*, which take values in a multiplier algebra $M(B)$ rather than the bounded operators on Hilbert space. The crossed product

[1] which is in fact the *raison d'étre* of the crossed product in the first place

comes from a *universal covariant homomorphism* (i_A, i_G) taking values in $M(A \times_\alpha G)$. We'll describe the universal property soon.

Ok, that's great, but often the crossed product is too big — the trade-off for having a nice universal property is that it's hard to "see" the elements of $A \times_\alpha G$. One particularly useful representation of the crossed product is the *regular representation*, analogous to (and generalizing) the regular representation of G. The image of $A \times_\alpha G$ under the regular representation is the *reduced crossed product* $A \times_{\alpha,r} G$. The reduced crossed product doesn't capture all the representation theory of the action (although it does encapsulate a lot of it) but the elements of $A \times_{\alpha,r} G$ are "easier to see" because they're (often quite familiar) operators on Hilbert space. The canonical covariant homomorphism associated to the regular representation is denoted (i_A^r, i_G^r).

The reduced crossed product encodes exactly those representations that factor through the regular representation. Consequently, there's a surjective homomorphism[2]

$$q^n : A \times_\alpha G \to A \times_{\alpha,r} G,$$

and in fact another definition of the reduced crossed product, more useful for our purposes, is as the quotient of $A \times_\alpha G$ by the kernel of the regular representation. Then the above surjection q^n is just the quotient map.

The crossed product carries a *dual coaction* $\widehat{\alpha}$ of G. Being a coaction[3], $\widehat{\alpha}$ is a certain type of homomorphism

$$\widehat{\alpha} : A \times_\alpha G \to M\big((A \times_\alpha G) \otimes C^*(G)\big).$$

The reduced crossed product carries its own version[4] $\widehat{\alpha}^n$ of the dual coaction, making the diagram

$$\begin{array}{ccc} A \times_\alpha G & \xrightarrow{\widehat{\alpha}} & M((A \times_\alpha G) \otimes C^*(G)) \\ {\scriptstyle q^n}\downarrow & & \downarrow{\scriptstyle q^n \otimes \mathrm{id}} \\ A \times_{\alpha,r} G & \xrightarrow[\widehat{\alpha}^n]{} & M((A \times_{\alpha,r} G) \otimes C^*(G)) \end{array}$$

commute.

So, the reduced crossed product $A \times_{\alpha,r} G$ is a C^*-algebra which comes equipped with certain other gadgets: a covariant homomorphism (i_A^r, i_G^r) and a coaction $\widehat{\alpha}^n$. The question underlying Landstad duality is:

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[2]and the reason for the superscript "n" will be given later

[3]more about these later

[4]and again the reason for "n" will come later

Question 1.3. What C^*-algebras arise in the form $A \times_{\alpha,r} G$?

As initially stated, that question is inappropriate: it asks for what algebras *coincide with* reduced crossed products. It's much more useful to ask

Question 1.4. What C^*-algebras are isomorphic to $A \times_{\alpha,r} G$ for some action (A, α) of G?

The most naive approach to the question would be to ask whether we can recover the action (A, α) from the C^*-algebra $A \times_{\alpha,r} G$ alone. Again, it would only be reasonable to ask for the action (A, α) up to isomorphism, rather than the specific action itself. However, it's probably unsurprising that the reduced crossed product C^*-algebra by itself is insufficient information to recover the action, even up to isomorphism. So, the next question is:

Question 1.5. Is there some extra information, besides $A \times_{\alpha,r} G$, that will allow us to recover the action (A, α) up to isomorphism?

Well, if we were also given the action (A, α) as extra information, that would suffice. But we want something a little less tautological.

Landstad's answer is:

Answer 1.6. Yes: in addition to $A \times_{\alpha,r} G$, it is enough to have the unitary homomorphism i_G^r and the dual coaction $\widehat{\alpha}^n$.

Note that we do not need to be given i_A^r, the other half of the canonical covariant homomorphism.

Thus, Landstad duality tells us that we can recover (A, α) up to isomorphism from the data $(A \times_{\alpha,r} G, \widehat{\alpha}^n, i_G^r)$. But we've see that it's quite cumbersome to make precise what "up to isomorphism" means. Here is where category theory can help: it turns out that there's a category of actions and a category of normal[5] coactions, and we can enhance the latter category so that the above data gives an equivalence between these categories, and moreover from this we can recover Landstad duality, plus more.

What are the categories? We start with

Definition 1.7. The *basic C^*-category* $\mathbf{C}^*$ has:

(i) objects: C^*-algebras;
(ii) morphisms: nondegenerate homomorphisms into multiplier algebras.

[5]more about what this "normal" is later

In more detail: if A and B are objects in $\mathbf{C}^*$ (i.e., they are C^*-algebras), then a *morphism* $\phi : A \to B$ in $\mathbf{C}^*$ is a nondegenerate homomorphism $\phi : A \to M(B)$. Here *nondegenerate* means $\phi(A)B = B$. It turns out that the naive idea of using ordinary homomorphisms between the C^*-algebras themselves is too restrictive. Of course, one should check that $\mathbf{C}^*$ really is a category, i.e., that it has identity morphisms (obvious) and that morphisms can be composed (not so obvious, but not deep).

A fundamental property of the basic category is that its isomorphisms are familar:

Lemma 1.8. *A morphism $\phi : A \to B$ in the category $\mathbf{C}^*$ is an isomorphism in $\mathbf{C}^*$ if and only if ϕ maps A into B and is an isomorphism of C^*-algebras in the usual sense.*

We need equivariant versions of the basic category; here is the appropriate version for actions:

Definition 1.9. The category $\mathbf{C^*act}$ has:

(i) objects: actions of G;
(ii) morphisms: equivariant morphisms in $\mathbf{C}^*$.

In more detail: if (A, α) and (B, β) are objects in $\mathbf{C^*act}$, then a *morphism* $\phi : (A, \alpha) \to (B, \beta)$ in $\mathbf{C^*act}$ is a morphism $\phi : A \to B$ in $\mathbf{C}^*$ that is $\alpha - \beta$ *equivariant* in the sense that

$$\phi \circ \alpha_s = \beta_s \circ \phi \quad \text{for all} \quad s \in G.$$

The theory of coactions is parallel to that for actions: the *crossed product* of a coaction (A, δ) is a C^*-algebra $A \times_\delta G$ whose representations are the same as the covariant representations of (A, δ), and more generally for the homomorphisms. To explain further we'll resort to a little bit of hand-waving:

When G is abelian, a coaction of G corresponds to an action of the dual group $\widehat{G}$. When G is nonabelian, a coaction of G is to be regarded as an action of the nonexistent dual group. Now, an action (A, α) of G corresponds to a certain type of *comodule* structure

$$\widetilde{\alpha} : A \to M(A \otimes C_0(G) = C_0(G, A)),$$

which, after $A \otimes C_0(G)$ has been identified with $C_0(G, A)$ and the multiplier algebra with $C_b(G, M^\beta(A))$ (the superscript "β" means $M(A)$ has the strict topology), is given by

$$\widetilde{\alpha}(a)(s) = \alpha_{s^{-1}}(a) \quad \text{for} \quad a \in A, s \in G.$$

Of course the same is true for actions of $\widehat{G}$: they correspond to certain comodule structures $A \to M(A \otimes C_0(\widehat{G}))$. The Fourier transform on G takes $C^*(G)$ isomorphically onto $C_0(\widehat{G})$. Thus, an action of $\widehat{G}$ corresponds to a certain comodule structure

$$A \to M(A \otimes C^*(G)),$$

and abstracting this structure so that all mention of $\widehat{G}$ has been expunged gives rise to the definition of a coaction:

Definition 1.10. A *coaction* of G on a C^*-algebra A is a monomorphism

$$\delta : A \to A \otimes C^*(G) \quad \text{in} \quad \mathbf{C}^*$$

satisfying

(i) $\overline{\operatorname{span}}\{\delta(A)(1 \otimes C^*(G))\} = A \otimes C^*(G)$;
(ii) $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta_G) \circ \delta$, where $\delta_G : C^*(G) \to C^*(G) \otimes C^*(G)$ is the morphism in $\mathbf{C}^*$ corresponding to the strictly continuous unitary homomorphism

$$s \mapsto s \otimes s$$
$$G \to M(C^*(G) \otimes C^*(G)).$$

(i) is a technical condition that makes crossed-product duality work[6], and (ii) is a kind of "co-associativity" corresponding to the property that, when G is abelian, an action of $\widehat{G}$ on A is a *homomorphism* from $\widehat{G}$ to the automorphism group $\operatorname{Aut} A$. It's important to observe that δ_G itself is a coaction of G on $C^*(G)$, called the *canonical coaction.*

Definition 1.11. The category $\mathbf{C}^*$**coact** has:

(i) objects: coactions of G;
(ii) morphisms: equivariant morphisms in $\mathbf{C}^*$.

In more detail: if (A, δ) and (B, ε) are objects in $\mathbf{C}^*$**coact**, then a *morphism* $\phi : (A, \delta) \to (B, \varepsilon)$ in $\mathbf{C}^*$**coact** is a morphism $\phi : A \to B$ in $\mathbf{C}^*$ that is $\delta - \varepsilon$ *equivariant* in the sense that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\delta} & A \otimes C^*(G) \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi\otimes\mathrm{id}} \\ B & \xrightarrow[\varepsilon]{} & B \otimes C^*(G) \end{array}$$

commutes in $\mathbf{C}^*$, which makes sense because

$$\phi \otimes \mathrm{id} : A \otimes C^*(G) \to B \otimes C^*(G)$$

is a morphism in $\mathbf{C}^*$.

[6] and used to be called *nondegeneracy* of the coaction

It's not so easy to describe the covariant homomorphisms of a coaction, and again we'll resort to hand-waving. As a warmup, let's rephrase covariance for actions: a covariant homomorphism (π, u) of an action (A, α) in a multiplier algebra $M(B)$ can be equivalently regarded as a morphism $\pi : (A, \alpha) \to (B, \operatorname{Ad} u)$ in $\mathbf{C^*act}$. Here $\operatorname{Ad} u : G \to \operatorname{Aut} B$ is the *inner action* implemented by the unitary homomorphism $u : G \to M(B)$.

By the universal property of the group C^*-algebra, u corresponds to a morphism[7] $u : C^*(G) \to B$ in $\mathbf{C^*}$. If G is abelian and α is an action of $\widehat{G}$, then a covariant homomorphism involves a morphism $u : C^*(\widehat{G}) \to B$ in $\mathbf{C^*}$. Fourier transforming, we get a morphism $\mu : C_0(G) \to B$ in $\mathbf{C^*}$, which implements an *inner coaction* of G on B. Generalizing to (possibly) nonabelian groups G, a covariant homomorphism of a coaction (A, δ) can be defined as a pair (π, μ), where $\mu : C_0(G) \to A$ is a morphism in $\mathbf{C^*}$ and $\pi : (A, \delta) \to (B, \operatorname{Ad} \mu)$ is a morphism in $\mathbf{C^*coact}$. There is a *universal covariant homomorphism* (j_A, j_G) of the coaction (A, δ) in $M(A \times_\delta G)$.

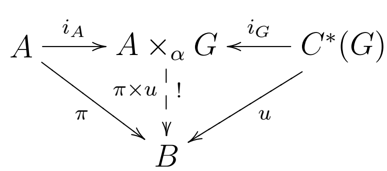
As alluded to before, in fact there are various flavors of coactions, and we want the *normal* ones, which means that the canonical morphism $j_A : A \to A \times_\delta G$ is injective; this makes life significantly easier.

Definition 1.12. The category $\mathbf{C^*coact^n}$ is the full subcategory of $\mathbf{C^*coact}$ whose objects are the normal coactions of G.

Now, to explain the *universal property* of crossed products, let's start with actions: given a covariant homomorphism (π, u) of an action (A, α) in $M(B)$, there is a unique morphism

$$\pi \times u : A \times_\alpha G \to B \quad \text{in} \quad \mathbf{C^*},$$

called the *integrated form of* (π, u), making the diagram



commute in $\mathbf{C^*}$. We need this universal property to describe what the crossed-product functor does to morphisms: given a morphism $\phi : (A, \alpha) \to (B, \beta)$ in $\mathbf{C^*act}$, there is a morphism

$$\phi \times G = (i_B \circ \phi) \times i_G : A \times_\alpha G \to B \times_\beta G \quad \text{in} \quad \mathbf{C^*}.$$

[7] and here we've made a choice to abuse notation by using u for two different maps; it's a fairly common trade-off

Similarly[8], the universal property for crossed products of coactions is the following: given a covariant homomorphism (π, μ) of a coaction (A, δ) in $M(B)$, there is a unique *integrated form*

$$\pi \times \mu : A \times_\delta G \to B \quad \text{in} \quad \mathbf{C}^*$$

making the diagram

$$\begin{array}{ccccc} A & \xrightarrow{j_A} & A \times_\delta G & \xleftarrow{j_G} & C_0(G) \\ & \searrow^{\pi} & {\scriptstyle \pi\times\mu}\downarrow{\scriptstyle !} & \swarrow_{\mu} & \\ & & B & & \end{array}$$

commute in $\mathbf{C}^*$. Now we can say what the crossed-product functor for coactions does to morphisms: given a morphism $\phi : (A, \delta) \to (B, \varepsilon)$ in $\mathbf{C}^*\mathbf{coact}$, there is a morphism

$$\phi \times G = (j_B \circ \phi) \times j_G : A \times_\delta G \to B \times_\varepsilon G \quad \text{in} \quad \mathbf{C}^*.$$

The crossed-product functor for actions has a "reduced" version: on objects, it does the obvious thing:

$$(A, \alpha) \mapsto A \times_{\alpha,r} G.$$

What about morphisms? Given a morphism $\phi : (A, \alpha) \to (B, \beta)$ in $\mathbf{C}^*\mathbf{act}$, we need a morphism $\phi \times_r G : A \times_{\alpha,r} G \to B \times_{\beta,r} G$ in $\mathbf{C}^*$ between the reduced crossed products. It's natural to look for it in a commutative diagram of the form

$$\begin{array}{ccc} A \times_\alpha G & \xrightarrow{\phi\times G} & B \times_\beta G \\ {\scriptstyle q^n_A}\downarrow & & \downarrow{\scriptstyle q^n_B} \\ A \times_{\alpha,r} G & \underset{\phi\times_r G}{\overset{!}{\dashrightarrow}} & B \times_{\beta,r} G. \end{array}$$

It turns out that this does uniquely determine a morphism $\phi \times_r G$, i.e., the kernel of $q^n_B \circ (\phi \times G)$ contains the kernel of q^n_A.

So now we have a reduced-crossed-product functor

$$\begin{aligned} (A, \alpha) &\mapsto A \times_{\alpha,r} G \\ \phi &\mapsto \phi \times_r G \\ \mathbf{C}^*\mathbf{act} &\to \mathbf{C}^*. \end{aligned}$$

But in Theorem 1.2 we need this functor to take values in an "enhanced" category, not just $\mathbf{C}^*$. Namely, we needed the image of an object (A, α) to include not only the reduced crossed product, but also

[8]more typically, the *cognoscenti* would say "dually"

the dual coaction $\widehat{\alpha}^n$ and the canonical map i_G^r. It's not hard to construct "with bare hands" a suitable category for this, but it turns out that category theorists have already done it abstractly. Well, actually we need to proceed in two steps here: first, we include the dual coaction. So, instead of taking values in $\mathbf{C^*}$, as an intermediate step we require the crossed-product functor to take values in $\mathbf{C^*coact}$:

$$(A,\alpha) \mapsto (A \times_{\alpha,r} G, \widehat{\alpha}^n).$$

As usual with functors, we should check the morphisms: if $\phi : (A,\alpha) \to (B,\beta)$ in $\mathbf{C^*act}$, does $\phi \times_r G$ give a morphism in $\mathbf{C^*coact}$? In other words, is it $\widehat{\alpha}^n - \widehat{\beta}^n$ equivariant? Unsurprisingly, the answer is affirmative, so we have a functor $\mathbf{C^*act} \to \mathbf{C^*coact}$. Actually, it turns out that dual coactions on reduced crossed products are automatically normal, so in fact we have a functor

$$\mathbf{C^*act} \to \mathbf{C^*coact^n}.$$

But we still need our functor to do one more job: keep track of i_G^r. Well, the unitary homomorphism $i_G^r : G \to M(A \times_{\alpha,r} G)$ corresponds to a morphism[9]

$$i_G^r : C^*(G) \to A \times_{\alpha,r} G \quad \text{in} \quad \mathbf{C^*},$$

which turns out to have a special property. To say what that property is, we need the actual definition of the dual coaction $\widehat{\alpha}^n$: it is the morphism from $A \times_{\alpha,r} G$ to $(A \times_{\alpha,r} G) \otimes C^*(G)$ in $\mathbf{C^*}$ determined by the covariant homomorphism

$$\big(i_A^r \otimes 1, i_G^r \otimes \mathrm{id}\big)$$

of (A,α) in $M((A \times_{\alpha,r} G) \otimes C^*(G))$, i.e.,

$$\widehat{\alpha}^n(i_A^r(a)) = i_A^r(a) \otimes 1 \quad \text{for} \quad a \in A \tag{1.2}$$

$$\widehat{\alpha}^n(i_G^r(s)) = i_G^r(s) \otimes s \quad \text{for} \quad s \in G.. \tag{1.3}$$

The important observation here is that (1.3) implies that the morphism $i_G^r : C^*(G) \to A \times_{\alpha,r} G$ is $\delta_G - \widehat{\alpha}^n$ equivariant, i.e., gives a morphism

$$i_G^r : (C^*(G), \delta) \to (A \times_{\alpha,r} G, \widehat{\alpha}^n)$$

in $\mathbf{C^*coact}$. But there's a subtlety: i_G^r is *not* a morphism in the subcategory $\mathbf{C^*coact^n}$, for the simple reason that the canonical coaction δ_G is typically not normal. In fact, δ_G is normal exactly when the group

[9] again notice the abuse of notation

G is amenable[10]. More generally, the dual coaction $(A \times_\alpha G, \widehat{\alpha})$ is normal if and only if $A \times_\alpha G = A \times_{\alpha,r} G$; this happens if G is amenable, and in some other special cases, but not always.

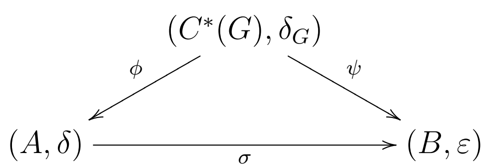
Anyway, we now have a category, namely $\mathbf{C^*coact}$, and a specific object in that category, namely $(C^*(G), \delta_G)$, and we are considering morphisms, of the form i_G^r, from that object into a particular subcategory, namely $\mathbf{C^*coact^n}$. We recognize our friend the comma category

$$(C^*(G), \delta_G) \downarrow \mathbf{C^*coact^n},$$

with

(i) objects: morphisms in $\mathbf{C^*coact}$ from $(C^*(G), \delta_G)$ to objects in $\mathbf{C^*coact^n}$;

(ii) morphisms: morphisms σ in $\mathbf{C^*coact}$ making a diagram of the form



commute.

An object in the comma category $(C^*(G), \delta_G) \downarrow \mathbf{C^*coact^n}$ is officially a pair $((A, \delta), \phi)$, but we'll simplify the notation by writing the object as a triple (A, δ, ϕ) instead of a pair.

Thus, we now see that, given an action (A, α), the triple $(A \times_{\alpha,r} G, \widehat{\alpha}^n, i_G^r)$ is an object in $(C^*(G), \delta_G) \downarrow \mathbf{C^*coact^n}$, and so finally we can understand the functor in Theorem 1.2.

Well, that was a long road, but we can now reap some reward for our effort. First, to discuss the reduced-crossed-product functor

$$\mathbf{C^*act} \to (C^*(G), \delta_G) \downarrow \mathbf{C^*coact^n}$$

further, let's give it a name: RCP. Being a category equivalence, it has the following properties:

(i) it is *essentially surjective* on objects, i.e., every object (B, δ, u) in $(C^*(G), \delta_G) \downarrow \mathbf{C^*coact^n}$ is isomorphic to one in the image of RCP;

[10]and we won't go into the precise definition of amenability here, except to say that it's equivalent to $C^*(G) = C_r^*(G)$, and is a good thing, enjoyed by, for example, abelian and compact groups, but not by the free group on two generators — which leads to the Banach-Tarski paradox, but that's another story, as Kipling would say

(ii) it is *full and faithful*, i.e., for any two objects (A,α) and (B,β) in $\mathbf{C^*act}$, the map

$$\begin{aligned}\phi \mapsto \phi \times_r G : \operatorname{Hom}\big((A,\alpha),(B,\beta)\big)\\ \to \operatorname{Hom}\big((A\times_{\alpha,r} G, \widehat{\alpha}^n, i_G^r),(B\times_{\beta,r} G, \widehat{\beta}^n, i_G^r)\big)\end{aligned}$$

is bijective.

We can now see how the properties of RCP allow us to recover Theorem 1.1 from Theorem 1.2: in Theorem 1.1, (i) follows from essential surjectivity on objects, and (ii) from fullness. Let's go through the routine verification of those statments:

Deducing Theorem 1.1 *from Theorem* 1.2. (i) First suppose we have an isomorphism $\theta : B \to A\times_{\alpha,r} G$. Then the diagrams

$$\begin{array}{ccc} G & \overset{u}{\dashrightarrow} & M(B) \\ & \searrow^{i_G^r} & \downarrow \cong \ \overline{\theta} \\ & & M(A\times_{\alpha,r} G) \end{array}$$

and

$$\begin{array}{ccc} B & \overset{\delta}{\dashrightarrow} & M(B\otimes C^*(G)) \\ \theta \downarrow \cong & & \cong \downarrow \overline{\theta\otimes\mathrm{id}} \\ A\times_{\alpha,r} G & \xrightarrow[\widehat{\alpha}^n]{} & M((A\times_{\alpha,r} G)\otimes C^*(G)), \end{array}$$

where $\overline{\theta} : M(B) \to M(A\times_{\alpha,r} G)$ is the canonical extension of θ to multipliers[11], and similarly for $\overline{\theta\otimes\mathrm{id}}$, can obviously be completed, giving the maps u and δ with the required properties (and the coaction δ is normal because it's isomorphic to the normal coaction $\widehat{\alpha}^n$).

Conversely, if such u and δ exist, then (B,δ,u) is an object in $(C^*(G),\delta_G)\downarrow \mathbf{C^*coact^n}$, and so is isomorphic to on in the image of RCP, and in particular B is isomorphic to a reduced crossed product.

(ii) Given the existence of u and δ, by essential surjectivity on objects we have an isomorphism

$$\theta : (B,\delta,u) \to (A\times_{\alpha,r} G, \widehat{\alpha}^n, i_G^r) \quad \text{in} \quad (C^*(G),\delta_G)\downarrow \mathbf{C^*coact^n},$$

i.e., the isomorphism $\theta : B \to A\times_{\alpha,r} G$ is $\delta - \widehat{\alpha}^n$ equivariant and satisfies $\theta\circ u = i_G^r$. Moreover, if we also have (C,β) and σ, then the

[11] whose existence is vouchsafed by nondegeneracy

commutative (by construction!) diagram

$$\begin{array}{ccc} (B,\delta,u) & \xrightarrow[\cong]{\theta} & (A\times_{\alpha,r}G,\widehat{\alpha}^n,i_G^r) \\ & \underset{\sigma}{\overset{\cong}{\searrow}} & \Big\downarrow{\scriptstyle \sigma\circ\theta^{-1}} \\ & & (C\times_{\beta,r}G,\widehat{\beta}^n,i_G^r) \end{array}$$

in $(C^*(G),\delta_G)\downarrow \mathbf{C^*coact^n}$ implies, by fullness, that there is a morphism $\phi:(A,\alpha)\to(C,\beta)$ in $\mathbf{C^*act}$ such that

$$\sigma\circ\theta^{-1}=\phi\times_r G,$$

so that

$$(\phi\times_r G)\circ\theta=\sigma,$$

and moreover ϕ is an isomorphism because $\phi\times_r G$ is, since RCP is an equivalence. $\square$

Actually, as a lagniappe, we start to see how careful study of the category theory can improve the result: the isomorphism $\sigma:(A,\alpha)\to(C,\beta)$ is unique, by faithfulness of the functor RCP.

And one more lagniappe: Steve and I were not able to prove the full version of Theorem 1.1 until we attacked it using the categorical approach (the uniqueness clause was originally complicated by something involving the regular representation). Somehow, the categorical structure forced us to verify enough facts that we were finally able to prove Theorem 1.1 in its present form. This was a pleasant surprise, giving an unexpected reward for using the categorical perspective.

2. Unity of language

Now we'll give another example of turning a characterization into a categorical equivalence, and we'll see that the category theory lets us bring into high relief the parallelism with Landstad duality for actions, thus affording a unity of language.

First we need to "dualize" (sorry) dual coactions: given a coaction (A,δ), the *dual action* $\widehat{\delta}$ of G on the crossed product $A\times_\delta G$ has automorphisms $\widehat{\delta}_s\in\operatorname{Aut}A\times_\delta G$ determined by the covariant homomorphisms

$$(j_A,j_G\circ\mathrm{rt}_s)$$

of (A,δ) in $M(A\times_\delta G)$, where "rt" is the action of G on $C_0(G)$ by right translation.

The following theorem originally appeared in [11, Theorem 3.3], and is the dual version of Theorem 1.1, characterizing crossed products by coactions rather than actions. We've modified the original statement considerably, using modern notation and terminology, and structuring it to be parallel to Theorem 1.1.

Theorem 2.1. *Let B be a C^*-algebra and G a locally compact group. Then*

(i) *B is isomorphic to a crossed product $A \times_\delta G$ by a coaction δ of G on a C^*-algebra A if and only if there are an action α of G on B and a $\mathrm{rt} - \alpha$ equivariant morphism $\mu : C_0(G) \to B$.*

(ii) *Moreover, if the above conditions hold then an isomorphism $\theta : B \to A \times_\delta G$ can be chosen to be $\alpha - \widehat{\delta}$ equivariant and to satisfy $\theta \circ \mu = j_G$, and if (C, ε) is another action and $\sigma : B \to C \times_\varepsilon G$ is an $\alpha - \widehat{\varepsilon}$ equivariant isomorphism such that $\sigma \circ \mu = j_G^\varepsilon$, then there is an isomorphism $\phi : (A, \delta) \to (C, \varepsilon)$ such that $(\phi \times G) \circ \theta = \sigma$.*

And now we'll state the categorical version, which Steve, Iain, and I proved a couple of years ago.

Theorem 2.2 ([7, Theorem 4.2])**.** *The functor*

$$(A, \delta) \mapsto (A \times_\delta G, \widehat{\delta}, j_G)$$
$$\phi \mapsto \phi \times G$$

is an equivalence from the category $\mathbf{C^*coact^n}$ *to the comma category* $(C_0(G), \mathrm{rt}) \downarrow \mathbf{C^*act}$.

Actually, [7, Theorem 4.2] is phrased in terms of *reduced coactions*, which are just like (full) coactions but use $C_r^*(G)$ rather than $C^*(G)$. It turns out that this causes no confusion or difficulty, because there is more-or-less complete freedom of choice between reduced and full coactions [12], so the preceding theorem follows immediately from the one appearing in [7].

Anyway, it's clear that Theorems 1.2 and 2.2 are dual versions of a common phenomenon, illustrating how categorical techniques provide a unity of language.

3. Improving results

Now we'll see how paying attention to the categorical perspective can lead to significant improvement of results. Again, the context is noncommutative duality, this time specifically the various types of coactions.

Recall that a coaction (A, δ) of G is called *normal* if the morphism $j_A : A \to A \times_\delta G$ is injective. This is admittedly a good thing, but we have to face up to the sad reality that not all coactions are normal, nor would we want to restrict our attention exclusively to the normal ones. After all, the canonical coaction $(C^*(G), \delta_G)$ is nonnormal in general — there are normal (and even reduced) versions of δ_G, but fortunately we don't need them.

To begin the discussion, we should talk a little bit about crossed-product duality (again, for a complete introduction, I recommend [3, Appendix A]). The reason why coactions were introduced in the first place is the following duality theorem for crossed products:

Theorem 3.1 ([4, Theorem 3.6])**.** *For any action* (A, α)*,*

$$(A \times_{\alpha,r} G) \times_{\widehat{\alpha}^n} G \cong A \otimes \mathcal{K}(L^2(G)).$$

In the above, $\mathcal{K}(L^2(G))$ denotes the compact operators on $L^2(G)$, and from now on we'll just write it as $\mathcal{K}$.

Thus, the crossed product by the dual coaction doesn't give back the original C^*-algebra A, but it comes close: $A \otimes \mathcal{K}$ has many of the same properties as A (and is called the *stabilization* of A), and in particular is Morita equivalent to A (so has the same primitive ideal space, representation theory, and K-theory, among other things).

Theorem 3.1 is phrased in terms of the reduced crossed product $A \times_{\alpha,r} G$ and the associated dual coaction $\widehat{\alpha}^n$. Somewhat later, Raeburn [13, Theorem 7] proved a version of crossed-product duality for the full crossed product:

Theorem 3.2 (Raeburn)**.** *For any action* (A, α)*,*

$$(A \times_\alpha G) \times_{\widehat{\alpha}} G \cong A \otimes \mathcal{K}.$$

There is a dual version for coactions. The following theorem was originally proved in [8, Theorem 8] using reduced coactions. The following version can be found in, for example, [3, Theorem A.69]

Theorem 3.3 (Katayama)**.** *For any normal coaction* (A, δ)*,*

$$(A \times_\delta G) \times_{\widehat{\delta},r} G \cong A \otimes \mathcal{K}.$$

For every coaction (A, δ), it is shown in [12] that there is a normal coaction (A^n, δ^n) and a morphism $q^n : (A, \delta) \to (A^n, \delta^n)$ in $\mathbf{C^*coact}$ such that

$$q^n \times G : A \times_\delta G \to A^n \times_{\delta^n} G$$

is an isomorphism. In particular, (A^n, δ^n, q^n) is an object in the comma category $(A, \delta) \downarrow \mathbf{C^*coact^n}$, called the *normalization* of (A, δ).

It turns out that there is another kind of coaction, which can be regarded as occupying the opposite end of the spectrum from the normal ones, for which crossed-product duality holds for the *full* dual crossed product. To describe them, it's useful to know that for any coaction (A, δ), [10] gives a surjection

$$\Phi : A \times_\delta G \times_{\widehat{\delta}} G \to A \otimes \mathcal{K},$$

called the *canonical surjection*. A coaction (A, δ) is called *maximal* if Φ is an isomorphism. For example, the dual coaction $(A \times_\alpha G, \widehat{\alpha})$ on the full crossed product by an action is always maximal [2, Proposition 3.4], and in particular the canonical coaction $(C^*(G), \delta)$ is maximal. The maximal coactions form a full subcategory $\mathbf{C^*coact^m}$ of $\mathbf{C^*coact}$.

It is shown in [2] that for any coaction (A, δ) there is a maximal coaction (A^m, δ^m) and a morphism $q^m : (A^m, \delta^m)$ in $\mathbf{C^*coact}$ such that

$$q^m \times G : A^m \times_{\delta^m} G \to A \times_\delta G$$

is an isomorphism. In particular, (A^m, δ^m, q^m) is an object in the comma category $\mathbf{C^*coact^m} \downarrow (A, \delta)$, called the *maximalization* of (A, δ).

For an action (A, α), the normalization map $q^n : A \times_\alpha G \to (A \times_\alpha G)^n$ has the same kernel as the regular representation [3, Proposition A.61], so the normalization is $(A \times_{\alpha,r} G, \widehat{\alpha}^n, q^n)$ (and this explains the notation $\widehat{\alpha}^n$ and q^n that we've been using). In particular, the normalization map for the canonical coaction $(C^*(G), \delta)$ is the regular representation $C^*(G) \to C^*_r(G)$.

Anyway, it is shown in [6] that the subcategories $\mathbf{C^*coact^m}$ and $\mathbf{C^*coact^n}$ of $\mathbf{C^*coact}$ are coreflective and reflective, respectively, and have all the properties mentioned in [1]. Thus we can apply all the results from [1]. In particular, we have the *maximal-normal equivalence*:

Theorem 3.4 ([6, Corollary 3.4])**.** *The restriction*

$$\mathrm{Nor}\,|_{\mathbf{C^*coact^m}} : \mathbf{C^*coact^m} \to \mathbf{C^*coact^n}$$

is an equivalence of categories.

This allowed us to prove a version of Landstad duality for coactions and full crossed products:

Theorem 3.5. [6, Theorem 5.1] *The full-crossed-product functor*

$$(A, \alpha) \mapsto (A \times_\alpha G, \widehat{\alpha}, i_G)$$
$$\phi \mapsto \phi \times G$$

is an equivalence from $\mathbf{C^*act}$ *to the comma category* $(C^*(G), \delta_G) \downarrow \mathbf{C^*coact^m}$.

In [6] we were able to show that Theorem 3.5 follows almost immediately from Theorems 1.2 and 3.4, once we proved that the composition of the full-crossed-product functor followed by normalization is naturally isomorphic to the reduced-crossed-product functor. There was a little bit of extra work to do, mainly to promote the maximal-normal equivalence to the comma categories $(C^*(G), \delta_G) \downarrow \mathbf{C^*coact^m}$ and $(C^*(G), \delta_G) \downarrow \mathbf{C^*coact^n}$.

Thus, paying attention to the categorical perspective allowed us to improve the result Theorem 1.2 by broadening Landstad duality to a version for full crossed products.

But in addition to this, the extra attention paid to the category theory in [1], in particular to the theory of adjoint functors, allows us to improve on the results in [6]. It transpires that maximalizations $q^m : (A^m, \delta^m) \to (A, \delta)$ are automatically surjective, hence epimorphisms in $\mathbf{C^*coact}$. The following two results are immediate consequences (by [1, Corollaries 7.7 and 7.8]):

Theorem 3.6. *The maximalization and normalization functors are faithful, so that if $\phi, \psi : (B, \varepsilon) \to (A, \delta)$ are morphisms in $\mathbf{C^*coact}$ such that*

$$\phi^m = \psi^m : (B^m, \varepsilon^m) \to (A^m, \delta^m),$$

then we must have $\phi = \psi$, and similarly $\phi^n = \psi^n$ implies $\phi = \psi$.

Theorem 3.7. *The normalization map $q^n : (A, \delta) \to (A^n, \delta^n)$ is a monomorphism in $\mathbf{C^*coact}$, so that if $\phi, \psi : (B, \varepsilon) \to (A, \delta)$ are morphisms in $\mathbf{C^*coact}$ such that*

$$q^n \circ \phi = q^n \circ \psi,$$

then we must have $\phi = \psi$.

Neither of these results were noticed in [6], and it seems to us that they would be difficult to prove without appealing to the categorical techniques. Theorem 3.7 in particular is non-intuitive, because the morphism $q^n : A \to A^n$ is certainly not usually a monomorphism in $\mathbf{C^*}$.

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