

Representation Dimension

Context/Descriptive/Verbal: Explorations based on the structure of a real-world, scientific or engineering context allow for descriptions of the context, direct references to observable entities in the system and development of relationships between them. Subsequent investigation of mathematical structure can flow from these contextual representations. Many aspects of a system may not be fully derived from context alone, thus generating a need for more abstract representations. Although connecting all representations is a key feature, the connections to the context warrant special emphasis, and results from other representations should be re-interpreted in terms of the context. Interpretation of parameters and attention to units are two particularly powerful tools within this representation.

Graph: Graphs visually capture large amounts of information about the relationship between the values of a relation so that many features of the covariation between the two quantities of the function may be observed quickly in gestalt. The graph is often the context where students' impoverished covariational reasoning abilities are initially revealed (Carlson et al., 2002). Thompson's work (2002) suggests that initially exercising students' dynamic imagery of two varying quantities in the phenomenon itself, especially connected to kinesthetic experience, helps students develop a mental model of the covariation.

Numbers/Numerical/Table: Numerical quantities often arise directly from a context by making actual measurements. Although tables may also be generated by a formula if that has been developed first, building the formula by recognizing numerical values in the table supports students in acquiring a meaningful understanding of the formula. The variables in the formula then emerge as a means of generalizing the patterns that students have already observed. Algebra then generalizes arithmetic which is tied to meaningful relationships in the context. Data in a table may also be used to examine and compute exact measurements for constructs (e.g., rate, average rate) that convey specific information about the covarying quantities at a point or over specific intervals of the function's domain.

Formula/Equation/Symbolic/Algebraic: Formulas often arise as generalized characterizations of computations performed or of patterns observed. Formulas then allow for powerful symbolic manipulations and characterization of patterns.

Parallels with other related context: Often, one might think of one context and its representations through an analogy with another context. For example motion and ideas of position and speed is often used as a way to think about covariation in systems that don't involve that sort of motion (e.g., the rate at which water pressure varies with depth may be conceived analogically as a "speed").

Motion Map (only for position/velocity contexts): Positions are represented as point locations and velocities as vectors extending from those locations.

Elements of Covariational Reasoning

Conceptual/Scientific Model: Most scientific phenomena are understood through a conceptual model consisting of objects and relationships between them e.g., the Bohr model of the atom. The fit of the idealized components of the model to actual entities in the real world may vary. The model components may represent either observable or unobservable entities and relationships from the context. A carrying capacity in a population model, for example, might not be directly observable but rather conceived as a plausible feature of the situation and introduced as part of a model to explain certain patterns.

Scale or Analogue Model: An actual replica of a physical system may often be constructed when the actual system may be too large, expensive, delicate, etc. to work with directly. This is extremely common in engineering where prototypes may help develop and test designs and parameters.

Mathematical Content Dimension

Functional Dependence – Pointwise: A key part of exploring the structure of a system is understanding what quantities depend on other quantities. This may be focused on pointwise determinations considering single values at a time. If an individual is restricted to this sort of reasoning (an action view of function), they will have difficulty conceiving dynamic relationships in the system.

Functional Dependence – Across Time: Covariational reasoning depends on conceiving of the functional dependence “across time,” either by dynamically imagining running through a continuum of values and considering the resulting impact on a dependent variable, or by thinking of the impact of the dependence on entire sets or intervals of values at a time (requiring a process view of function). Mental Action 1 (MA1) from Carlson’s framework is a part of this sort of judgment of dependence “across time.”

Direction of Change: After conceiving of functional dependence across time, the first judgment one might make is the directionality of dependence. This is Mental Action 2 (MA2) in Carlson’s Framework.

Landmarks: An important part of the analysis of functional dependence is attending to certain configurations in the context where one type of variation changes to another type of variation. This often occurs at points that relate to important graphical features such as extrema, domain endpoints, inflection points, etc. (e.g., a maximum occurs when changing from increasing to decreasing). While these landmarks themselves are single points, they only become landmarks due to the variation of the function around that point. Thus their discernment and interpretation requires covariational reasoning about the corresponding variation.

Amounts of Change - Pointwise: An initial step toward unpacking rate of change and covariation is attending to the actual amounts of change in a situation between two possible configurations. Often the configurations are chosen in the analysis so that the changes are quite small, though many times they are determined from the nature of observation or data collection. Pointwise, we may focus on the amounts of change between only two configurations.

Amounts of Change – Across Time: Covariational reasoning involves making judgments about change across time. Considering amounts of change on contiguous or sliding intervals and coordinating how these changes vary across those intervals is Carlson’s Mental 3 (MA3). At this point, one may coordinate “speed-lengths” to make non-proportional judgments about the relationships between the quantities and their changes.

Elements of Covariational Reasoning

Amounts of Change – Parameterized: Individuals frequently reason or gesture as if the covariation between two quantities were parameterized by a third variable. Typically the parameter is time, but may be something else such as arc length in a graph. Parameterized reasoning at this level proceeds by advancing the parameter by some amount incrementally and comparing resulting changes in the two quantities.

Amounts of Change – Proportional: An intermediary between MA3 judgments and a quantitative conception rate involves conceiving of multiple pairs of changes representing the same rate, that is, reasoning proportionally about a constant rate. This may be abstracted to a representation of all possible pairs of changes in the same ratio.

Constant Rate: A constant rate is viewed proportionally as described above, but the quality being quantified is different; it is the rate rather than the changes.

Average Rate – Pointwise: When rate of change is not constant, we may still conceptualize an average rate of change, specifically, as a constant rate for a mentally auxiliary situation in which the same amount of change in both quantities is achieved. Conceiving of an average rate as the constant rate that accumulates the same amount as a changing rate over a given interval involves some notion (at least iconic) of instantaneous rate across time (see below).

Average Rate – Across Time: Coordinating the average rates described above across multiple intervals constitutes Carlson's Mental Action 4 (MA4)

Instantaneous Rate – Pointwise: Applying a limiting process to the average rate results in the instantaneous rate of change at a point. For this to be anything more than iconic requires a dynamic image of average rate across a diminishing interval size.

Instantaneous Rate – Across Time: Coordinating instantaneous rates across time constitutes Carlson's Mental Action 5 (MA5). The variation across time may be conceived as the derivative as a function of the independent variable, setting up an entirely new covariational relationship (between the instantaneous rate and the independent variable).

Instantaneous Rate – Parameterized: Like the parameterized version of MA3, this involves the advancement of a parameter (typically time or arc length) and making comparisons about the rates at which the other two quantities are changing.

Quantification Dimension

Calculational: Mathematical instruction, and thus understanding, often focuses on the numerical operations (calculations) to find a quantity, regardless of its interpretation.

Iconic: An object may be perceived without constructing it through a process from prior objects. Thus it cannot be unpacked into its constituent parts. An example of this is imagery is conceiving of speed as steepness in a graph without being able to further analyze its meaning in terms of actual changes on the graph.

Gross: Quantities may be considered, compared, etc. by experiential criteria and seriation. Often this is connected to some form of an iconic quantification.

Indirect: One may make judgments about one quantity based on actual quantification of another quantity. For example, one may compare two displacements starting from the same location, by quantifying only the ending positions. What is being quantified is position rather than displacement. This example is a subtle distinction since displacement and position are measuring similar qualities. More importantly, one may make judgments about a rate by quantifying changes in distance and time. Here the qualities being quantified are different in nature than the one being judged.

Extensive: Composing numerical elements from unitizing or segmenting is the basis of extensive quantification (Thompson, 1994). A quantity is composed of an object, a quality of the object, an appropriate unit or dimension, and a process by which to assign a numerical value to the quantity.

References

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