MAT 371
Chapter 5 – Study Guide

1. Read the first paragraph of Section 5.1.
   a. $P_1 = \{1, 1.5, 1.7, 2.9, 2.98, 3\}$ is a partition of ____________.
   b. Let $P_2 = \{1, 1.03, 1.5, e, 3\}$. Is P_1 a refinement of P_2? Is P_2 a refinement of P_1? Both? Or Neither?
   c. Find a partition that is a refinement of both P_1 and P_2.

2. Read the next paragraph of Section 5.1.
   a. Why is it necessary to suppose *f* is bounded?
   b. What is the meaning of $U(P, f)$ and $L(P, f)$ graphically?
   c. Suppose $v(t)$ is the velocity of an object at time *t*. What is the meaning of $U(P, f)$ and $L(P, f)$ in this context?

3. Read the definitions of $\overline{\int_a^b} f\ dx$ and $\underline{\int_a^b} f\ dx$.
   a. What is considered to vary in these definitions producing different values in order to consider an inf or a sup?
   b. Give an example of a function on an interval $[a,b]$ for which $\overline{\int_a^b} f\ dx \neq \underline{\int_a^b} f\ dx$.

4. Read Theorem 5.1.
   a. Draw a (large) picture to illustrate the proof of part i. Label everything.
   b. Read the proof to part ii. Why does this work, even if *P* and *Q* are completely different?
   c. Read the proof to part iii. What is the key idea of the proof?

5. Read Theorem 5.2.
   a. Let $f(x) = x^2$. Find a partition *P* of [–1,1] such that $U(P, f) - L(P, f) < 0.5$.
   b. Read the proof of the direction [$\Rightarrow$]. What are the big ideas in the proof? How is the supremum used (especially in conjunction with the ε)? What are the various partitions that show up in this direction of the proof and how are they used?
   c. Read the proof of the direction [$\Leftarrow$]. How are they able to conclude that $0 \leq \overline{\int_a^b} f\ dx - \underline{\int_a^b} f\ dx \leq \varepsilon$? Why does this complete the proof?

6. Read Theorem 5.3 and its proof. Carefully justify each equality and inequality in the final string of expressions at the end of the proof.

7. Read Theorem 5.4 and its proof.
   a. Why is it important to know that *f* is in fact uniformly continuous.
   b. Why can we claim that there are $t_i, s_i \in [x_{i-1}, x_i]$ such that $M_i(f) = f(t_i)$ and $m_i(f) = f(s_i)$? How is this result then used in the proof?
   c. Carefully justify each equality and inequality in the final string of expressions at the end of the proof.

8. Read the first paragraph of Section 5.3. How is a Riemann sum different from the sums used to define the Riemann integral earlier? Read the statements of Theorems 5.5, 5.6, and 5.7 (not the proofs). How do these relate Riemann sums to the Riemann integral?

9. Read Theorem 5.8 (The Fundamental Theorem of Calculus) and its proof. Draw a picture that illustrates how the Mean Value Theorem is used in the proof.

10. Read Theorem 5.13 (Taylor's Theorem).
   a. What does this have to do with Taylor series?
   b. Explain how this theorem is just a restatement of the Fundamental Theorem of Calculus for the case that $n = 0$.
   c. Set up and complete the details of the integration by parts used in the proof.
   d. Read the version of Taylor's theorem presented on page 163. How does it differ?
   e. Read the proof of the version on page 163. How does it use Theorem 5.15?
   f. Prove this result in the following way:
      i. Define $F(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!}(t)(x-t)^n + M(x-t)^{n+1}$ where *M* is defined so that $F(a) = f(x)$. Why is this possible?
      ii. Show that $F(a) = F(x)$. Then apply the Mean Value Theorem to the interval $[a, x]$.
      iii. Solve for M.
      iv. Does this proof look familiar?

11. Read Theorem 5.14.
    a. What name is commonly given to this theorem?
    b. Read the proof.
    c. Explain why the derivative involved in this theorem can be expressed as $\frac{d}{dx}\int_a^x f(t)dt = \lim_{h\to 0}\frac{\int_x^{x+h} f(t)dt}{h}$. Draw a picture illustrating the meaning of the integral in the numerator of the right side of this equation.
    d. A common visual interpretation of this expression is that the integral $\int_x^{x+h} f(t)dt$ represents a thin slice of area under the graph between *x* and $x+h$. As $h \to 0$, then this area becomes the 1-dimensional line of height $f(x)$. Thus $\frac{d}{dx}\int_a^x f(t)dt = f(x)$. Draw a picture illustrating this argument. What is its flaw?
    e. Another interpretation of the expression from Part 1 is:

        i. since *h* is small, the values of *f* on the interval $[x, x+h]$ are all approximately equal to $f(x)$,
        ii. the integral $\int_x^{x+h} f(t)dt$ is approximately the area of the rectangle with height $f(x)$ and width *h*, i.e., $\int_x^{x+h} f(t)dt = f(x)\bullet h$,
        iii. so the difference quotient is $\frac{f(x)\bullet h}{h} = f(x)$.

       Draw a picture of illustrating this argument. This is very close to a correct and rigorous argument, but it cheats a little bit. What is the flaw?
    f. Carefully justify each equality and inequality in the final string of expressions at the end of the proof in the book.

12. Read Theorem 5.18 (Change of Variable).
    a. What name is commonly given to this theorem in introductory calculus courses?
    b. Read the proof and identify the key ideas.