MAT 371
Section 0.5 Study Guide

Algebraic Properties of the Real Numbers

1. Read Theorem 0.19 Part i.
   a. Read the proof to Part i. What is the key idea of this proof?
   b. There is no mention of subtraction in the axioms and we have not defined the operation. Define explicitly what is meant by "$-x-y$" in this proof.
   c. The phrase "removing the parentheses and simplifying" involves the implicit application of several of the axioms for a complete ordered field. Write out a series of equalities so that each step involves the use of one axiom and identify the axiom for each step.

2. Read Theorem 0.19 Part ii.
   a. Read the proof to Part ii. This is a proof by contradiction. State what is being assumed, what contradiction is derived, and what is subsequently concluded.
   b. It is not stated, but each of the following facts are needed in this proof. Explain why each is true from the axioms.
      $-(-1)=1$ (Hint: Write the definition of $-(-1)$ then show 1 has this property.)
      $0\bullet x=0 \quad \forall x$ (Hint: Add both x and $-x$ to $0\bullet x$ then simplify using the axioms.)
   c. This proof explicitly uses the following facts without justification. Explain why each is true from the axioms and the two properties you established in Part b.
      $-0=0$ (Hint: Write the definition of -0. Then show 0 has that property.)
      $(-1)(-1)=1$ (Hint: Show that $(-1)(-1)$ has the property defining $-(-1)$.)

3. Read Theorem 0.19 Part iii and its proof.
   a. What is the key idea of this proof?
   b. Identify which axioms justify each claim made in the proof.

4. Read the definition of absolute value and Theorem 0.25 on p. 26. You will use these properties regularly throughout this course, especially the triangle inequality. Prove the triangle inequality more directly by first showing that $|a+b|^2 \le (|a|+|b|)^2$.

The Least Upper Bound Property and its Implications

5. Reread the definition of a least upper bound on p. 22.
   a. Give an example of a subset *A* of the real numbers whose l.u.b. is a member of *A*. Describe the set of all upper bounds of *A*.
   b. Give an example of a subset *B* of the real numbers whose l.u.b. is not a member of *B*. Describe the set of all upper bounds of *B*.
   c. Give an example of a subset *C* of the real numbers that does not have a l.u.b. Describe the set of all upper bounds of *C*. Can you create an example that has a different set of upper bounds?
   d. Give an example of a subset *D* of the rational numbers that is bounded from above but does not have a least upper bound.
   e. Give an example of a subset *E* of the rational functions (with "ordering by end behavior" as discussed in class) that is bounded from above but does not have a least upper bound.

6. Read Theorem 0.20 and its proof.
   a. Sketch a number line with a set *S* that is bounded from below. (Your diagram will be cleanest if you make *S* include only positive or only negative numbers.)
   b. What is the justification for the statement that $-s \le -M$?
   c. Label *T*, *B*, and $-B$ on your diagram then label examples of *C* and $-C$.
   d. What would you describe as the "key idea" of the proof? Based on that idea, reproduce your diagram then write down the entire proof without looking at your book.

7. Read Theorem 0.23 and its proof.
   a. Draw a number line marked with an arbitrary point $p \ge 1$. Add the set *A* to your diagram. It will also be convenient to indicate where the set $\{y : y = z^2 \text{ and } z \in A\}$ is located.
   b. Fill in the details that if $z > p$ and $p \ge 1$ then $z^2 > p^2 \ge p$. Why does this mean that *p* is an upper bound for *A*?
   c. Add $x = \sup A$ to your diagram. In deriving a contradiction to the case $x^2 < p$, the definition for δ probably seems to come out of the blue. To see where it comes from, we want to show there is a small amount δ that we can add to *x* and still be in the set *A*. In other words, we want $(x+\delta)^2 < p$. Expand this inequality and identify what makes solving for δ messy. Explain why replacing the δ^2 term with δ would make this easier, then explain why we can choose $\delta \le 1$ to achieve this. Finally, solve the inequality for δ and compare your result to the definition of δ given in the proof. In order to reproduce this entire portion of the proof (contradicting the case $x^2 < p$), what do you need to remember, that is, what is the key idea?
   d. Follow a similar process as you did in Part c, but this time to understand the definition of δ used in deriving a contradiction to the case $x^2 > p$.
   e. Read and explain the key idea for the proof of the case that $0 < p < 1$.

The Archimedean Property and its Implications

8. Read Theorem 0.21 and its proof. We will refer to this as the *Archimedean property*.
   a. The Archimedean property basically rules out the existence of infinite and infinitesimal numbers. Explain how.
   b. Describe the set *A* in words. Draw a number line with *x* labeled at some location, then add the elements of *A* to the diagram. As you answer the following questions, add the variables to your diagram.
   c. How do you know that *A* is bounded from above? Why does this imply that *A* has a least upper bound, n_0? Why does this mean that $n_0 - 1$ is not an upper bound for *A*?
   d. Can you prove that n_0 is an integer? Can you prove that $n_0 \le x$? Make sure your diagram reflect your answers.
   e. What is the purpose of *m* in the proof? Specifically, why not just use n_0? And how do you know that $x < m+1$?
   f. For the case that $A = \varnothing$, why does $B = Z$? Why is *B* bounded from below? Why is $z_0 - 1 \notin B$? Why is this a contradiction?

9. Does the set of rational numbers have the Archimedean property? Does the set of rational functions have the Archimedean property? If they do have the Archimedean property, explain why. If not, give an example of an element that is larger than all of the integers.

10. Read Theorem 0.22 and its proof. The choices to apply the Archimedean property to $\frac{1}{y-x}$ then to Nx probably seem to come completely out of the blue. These choices were actually carefully made as a result of advanced planning. Seeing how is crucial to understanding this proof, and the following questions should clarify this for you.
    a. Let *x* and *y* be real numbers with $x < y$ and write an inequality involving a rational number $\frac{p}{q}$ capturing what we need to prove. Multiply everything in your inequality by *q*, then explain why this means you want *q* to be large enough so that $q(y-x) > 1$. Explain how you can rewrite this inequality and use the Archimedean property to find such a *q*. What plays the role of *q* in the proof in the book?
    b. Now explain how you can use the Archimedean property again to find *p*. What plays the role of *p* in the proof in the book?