MAT 310
Sections 6.4 – 6.5 Study Guide

Recall that on the Poincaré disk lines are asymptotic if they get arbitrarily close together as they approach the boundary of the disk. Three lines that are pairwise asymptotic form an *ideal triangle* (see Figure 1). An ideal triangle is not a triangle since the lines do not intersect so there are no vertices. Nevertheless we sometimes think of an ideal triangle as having 3 vertices on the boundary of the Poincaré disk where the lines converge, or "at infinity" in our hyperbolic space. A 2/3-*ideal triangle* is a figure formed by a line asymptotic to two rays from a common endpoint (see Figure 2). We call such a figure a 2/3-ideal triangle since it has 2 of its 3 vertices "at infinity."

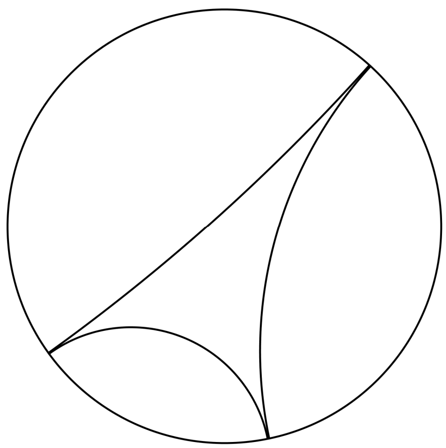


Figure 1.
Ideal Triangle (with all 3 vertices "at infinity")

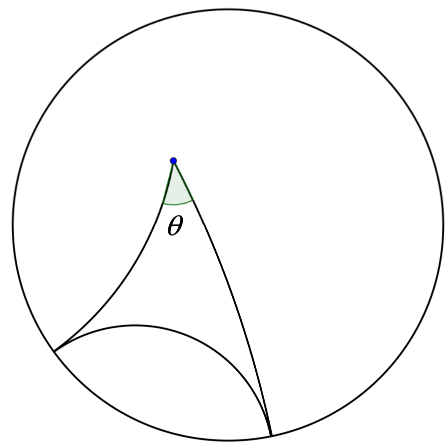


Figure 2.
2/3-Ideal Triangle (with 2 vertices "at infinity")

1. Discuss among your group the reason why any two 2/3-ideal triangles with the same angle are congruent.

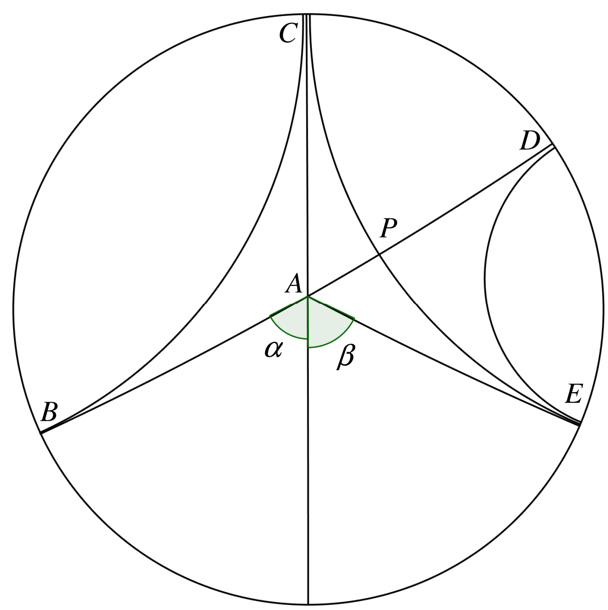
2. Suppose that our hyperbolic plane has radius ρ. Since all 2/3-ideal triangles with the same angle are congruent, we can define an area function:

   $A_\rho(\alpha)$ = the area of the 2/3-ideal triangle with *exterior* angle α.

   Show that the area function A_ρ is additive, that is,

   $$A_\rho(\alpha+\beta) = A_\rho(\alpha) + A_\rho(\beta).$$

   Hint: In the figure to the right, show that the area of $\triangle ADE$ is the sum of the areas of triangles $\triangle ABC$ and $\triangle ACE$ by showing that $\triangle PDE$ is congruent to $\triangle PBC$.



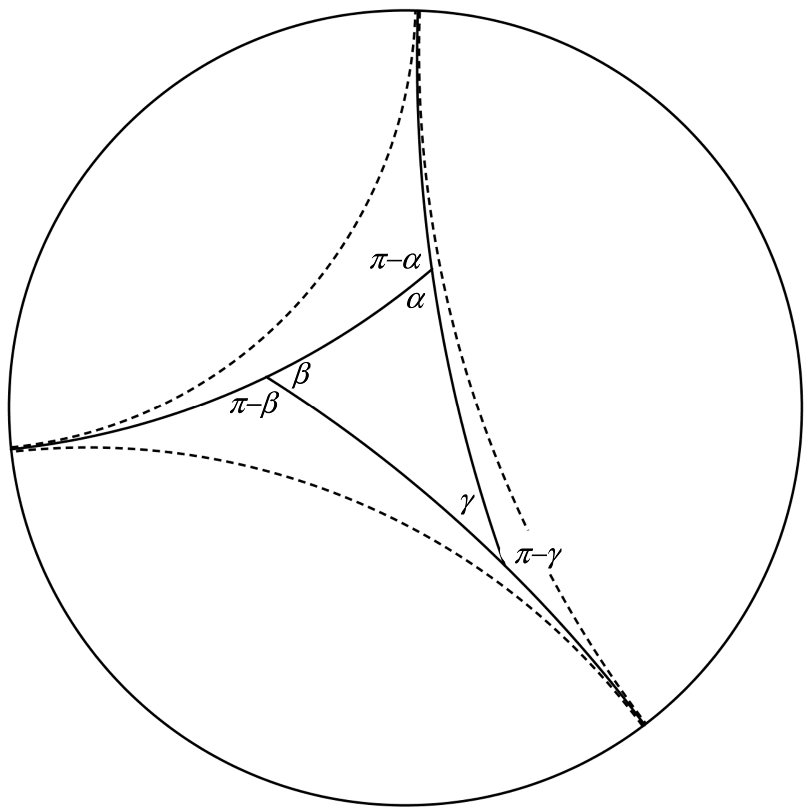
   Explain why this means the area function is linear, that is $A_\rho(\alpha) = \text{constant} \times \alpha$.

3. What is $A_\rho(0)$? What is $A_\rho(\pi)$? What kind of triangle is represented by each of these areas? (Remember that the input is the *exterior* angle.)

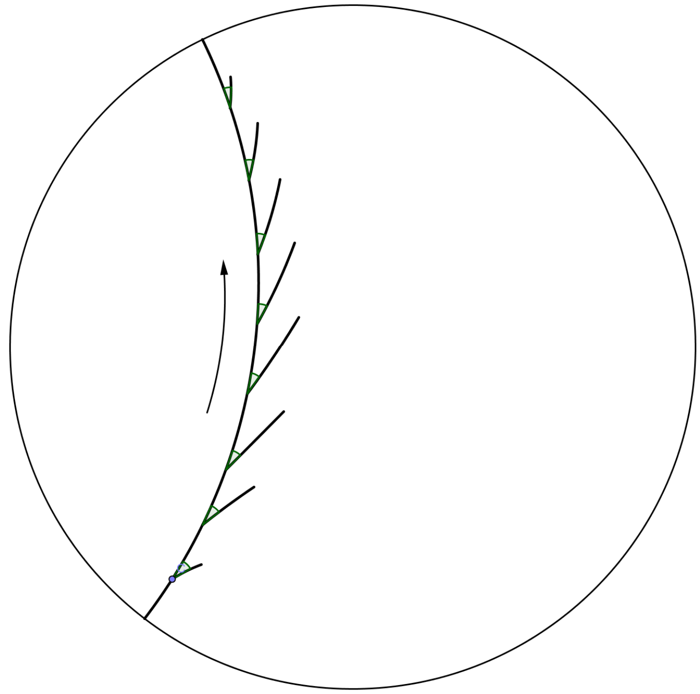
4. Explain why all ideal triangles on the same hyperbolic plane have the same area. Later we will see that the area of an ideal triangle is $\pi\rho^2$.

5. Find a formula for the area of a 2/3-ideal triangle with *exterior* angle α.

6. Using the diagram below, find a formula for the area of a hyperbolic triangle with *interior* angles α, β, and γ. What can you conclude about the sum of the angles of a triangle on the Poincaré disk?



7. Definition 6.5.1 in the book defines area to be a constant times the defect of the triangle. Explain how what you have done in Problem 6 shows that this is the only possible definition for area on the Poincaré disk.



Parallel Transport and Holonomy: Recall that given a line and a point not on the line in the Poincaré disk, there are infinitely many parallels. That is, parallels through a point are not unique. There is, however, a way to define parallels moving along lines. Moving along a line in the Poincaré disk involves a no-turning condition (an idea we will explore later when we discuss differential geometry). So holding a stick at a constant angle to our direction of movement means the stick is also not turning. The direction a vector (the stick) is pointing at the end of such a motion is called the *parallel transport* of the vector along the line. This concept can also be extended to motions on paths other than lines (again we will explore this when we discuss differential geometry).

The result of a parallel transport depends on the path taken. Additionally, if we complete a full loop, returning to our starting point, the parallel transported vector will generally point in a different direction than the original vector. The amount of rotation (measured in the same direction, clockwise or counterclockwise, as you traveled around the loop) as is called the *holonomy* of the path.

8. Find a formula that expresses the holonomy of a triangle on the Poincaré disk in terms of the *interior* angles, α, β, and γ.