MAT 310

Homework 13 - Solutions

1. Use spherical coordinates for a sphere of radius r for the following questions.
   a. Parameterize a path $\gamma:[0,1]\to M$ for a circle at latitude φ_0.

   The path is parameterized by $\gamma(t)=\begin{bmatrix} r\sin\varphi_0\cos t \\ r\sin\varphi_0\sin t \\ r\cos\varphi_0 \end{bmatrix}$ for $0\le t\le 2\pi$.

   b. Find the velocity $\gamma'(t)$ and acceleration $\gamma''(t)$ vectors for this path in $\mathbf{R}^3$. Explain why the velocity vector is intrinsic but the acceleration vector is extrinsic.

   The velocity and acceleration vectors are

   $$\gamma'(t)=\begin{bmatrix} -r\sin\varphi_0\sin t \\ -r\sin\varphi_0\cos t \\ 0 \end{bmatrix} \text{ and } \gamma''(t)=\begin{bmatrix} -r\sin\varphi_0\cos t \\ -r\sin\varphi_0\sin t \\ 0 \end{bmatrix}.$$

   The since the path lies in the sphere of radius r, the velocity vectors are always tangent to the surface, i.e., $\gamma'(t)\in T_{\gamma(t)}M$. This is intrinsic because it is completely detectable from within the sphere itself. The acceleration $\gamma''(t)$ is not generally tangent to the surface since it has a component required to simply stay on the surface of the sphere, thus is extrinsic.

   c. Find the covariant derivative $\nabla_{\gamma'(t)}\gamma'(t)$ by projecting the acceleration vector $\gamma''(t)$ onto the tangent plane c.

   Since the radius of the sphere is perpendicular to the tangent plane at each point, we can define the unit normal as $\mathbf{n}=\begin{bmatrix} \sin\varphi_0\cos t \\ \sin\varphi_0\sin t \\ \cos\varphi_0 \end{bmatrix}$. The projection of $\gamma''(t)$ onto the normal $\mathbf{n}$ is

   $$\begin{aligned} \operatorname{Proj}_{\mathbf{n}}\gamma''(t) &= \left(\gamma''(t)\bullet\mathbf{n}\right)\mathbf{n} \\ &= \left(-r\sin^2\varphi_0\cos^2 t - r\sin^2\varphi_0\sin^2 t\right)\mathbf{n} \\ &= \left(-r\sin^2\varphi_0\right)\begin{bmatrix} \sin\varphi_0\cos t \\ \sin\varphi_0\sin t \\ \cos\varphi_0 \end{bmatrix} \end{aligned}$$

Then covariant derivative $\nabla_{\gamma'(t)}\gamma'(t)$ is the projection of the standard derivative $\gamma''(t)$ onto the tangent plane $T_{\gamma(t)}M$

$$\begin{aligned}
\nabla_{\gamma'(t)}\gamma'(t) &= \mathrm{Proj}_{T_{\gamma(t)}M}\ \gamma''(t) \\
&= \gamma''(t) - \mathrm{Proj}_{\mathbf{n}}\ \gamma''(t) \\
&= \begin{bmatrix} -r\sin\varphi_0\cos t \\ -r\sin\varphi_0\sin t \\ 0 \end{bmatrix} + r\sin^2\varphi_0 \begin{bmatrix} \sin\varphi_0\cos t \\ \sin\varphi_0\sin t \\ \cos\varphi_0 \end{bmatrix} \\
&= r\sin\varphi_0 \begin{bmatrix} -\cos t\left(1-\sin^2\varphi_0\right) \\ -\sin t\left(1-\sin^2\varphi_0\right) \\ \sin\varphi_0\cos\varphi_0 \end{bmatrix} \\
&= r\sin\varphi_0\cos\varphi_0 \begin{bmatrix} -\cos t\cos\varphi_0 \\ -\sin t\cos\varphi_0 \\ \sin\varphi_0 \end{bmatrix}
\end{aligned}$$

d. Show that the covariant derivative $\nabla_{\gamma'(t)}\gamma'(t)$ is only zero at the poles and the equator. Explain what this means about geodesics on the sphere.

In order for $\nabla_{\gamma'(t)}\gamma'(t) = \mathbf{0}$, either $\sin\varphi_0 = 0$ or $\cos\varphi_0 = 0$. On the interval of possible values for φ_0 which is $[0,\pi]$, this happens when $\varphi_0 = 0,\ \pi/2$, or π.

At the north pole ($\varphi_0 = 0$) the path itself is constant, $\gamma(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ for all *t*. So the velocity $\gamma'(t) = 0$ and the acceleration is also zero. But since this is just a point and not a curve, it is not a geodesic, or "straight line" on the sphere. Likewise, at the south pole ($\varphi_0 = \pi$), the path is constant $\gamma(t) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$. At the equator, $\varphi_0 = \pi/2$, so $\gamma(t) = \begin{bmatrix} r\cos t \\ r\sin t \\ 0 \end{bmatrix}$ represents an actual path and satisfies the "no-turning condition" of $\nabla_{\gamma'(t)}\gamma'(t) = 0$. Thus the equator is a geodesic, or "straight line," on the sphere.

2. Consider the following coordinates for a cylinder of radius r

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = h$$

a. Describe what all paths look like on the cylinder parameterized by

$$x = r\cos(at)$$
$$y = r\sin(at)$$
$$z = bt + c$$

for different values of a, b, and c.

Since $\theta = at$, the parameter a is the angular frequency, determining the rate at which the path winds around the cylinder. Since $h = bt + c$, the parameters b and c determine the linear vertical motion. Specifically, b is the rate at which the path moves up the cylinder and c is the starting height on the cylinder (for $t = 0$). So the following paths are possible:

$a = 0$: A vertical line on the cylinder

$b = 0$: A circular path at a fixed height $h = c$

$a \neq 0$, $b \neq 0$: Spiral paths steeper for smaller values of a and larger values of b and more compressed for larger values of a and smaller values of b.

b. Show that these paths are all geodesics on the cylinder by showing that the covariant derivative is zero.

The standard derivatives are $\gamma'(t) = \begin{bmatrix} -ar\sin at \\ ar\cos at \\ b \end{bmatrix}$ and $\gamma''(t) = \begin{bmatrix} -a^2 r\cos at \\ -a^2 r\sin at \\ 0 \end{bmatrix}$. The normal vector at a point on this path just points in the direction of the horizontal components of the location, so $\mathbf{n} = \begin{bmatrix} \cos at \\ \sin at \\ 0 \end{bmatrix}$. We can immediately see that $\gamma''(t) = -a^2 r\mathbf{n}$, so the projection of $\gamma''(t)$ onto $\mathbf{n}$ is $\gamma''(t)$ itself. Thus the projection of $\gamma''(t)$ onto the tangent plane is the zero vector, so the covariant derivative is $\nabla_{\gamma'(t)}\gamma'(t) = \mathbf{0}$. Thus these paths are all geodesics.

3. The latitude of Tempe, AZ is 33°23' N. Determine the scalar speed (s) and the scalar intrinsic acceleration (a) for the path at this latitude. Recall that the rate at which v is turning is a/s. Determine this rate and multiply by the duration of one revolution to find the total amount rotation of the velocity vector.

From Problem 2, we know that the velocity is $\gamma'(t) = \begin{bmatrix} -r\sin\varphi_0 \sin t \\ -r\sin\varphi_0 \cos t \\ 0 \end{bmatrix}$, and the intrinsic acceleration is $\nabla_{\gamma'(t)}\gamma'(t) = r\sin\varphi_0\cos\varphi_0 \begin{bmatrix} -\cos t\cos\varphi_0 \\ -\sin t\cos\varphi_0 \\ \sin\varphi_0 \end{bmatrix}$. From this, we can compute the scalar speed is

$$\begin{aligned} s &= \sqrt{\gamma'(t)\bullet\gamma'(t)} \\ &= \sqrt{r^2\sin^2\varphi_0\sin^2 t + r^2\sin^2\varphi_0\cos^2 t} \\ &= \sqrt{r^2\sin^2\varphi_0} \\ &= r\sin\varphi_0 \end{aligned}$$

and the scalar intrinsic acceleration is

$$\begin{aligned} a &= \sqrt{\nabla_{\gamma'(t)}\gamma'(t)\bullet\nabla_{\gamma'(t)}\gamma'(t)} \\ &= r\sin\varphi_0\cos\varphi_0\sqrt{\cos^2 t\cos^2\varphi_0 + \sin^2 t\cos^2\varphi_0 + \sin^2\varphi_0} \\ &= r\sin\varphi_0\cos\varphi_0\sqrt{\cos^2\varphi_0 + \sin^2\varphi_0} \\ &= r\sin\varphi_0\cos\varphi_0. \end{aligned}$$

Then the rate at which the velocity vector is turning intrinsically is

$$\begin{aligned} \omega &= \frac{a}{s} \\ &= \frac{r\sin\varphi_0\cos\varphi_0}{r\sin\varphi_0} \\ &= \cos\varphi_0. \end{aligned}$$

The longitude is 33°23' (which is 33.3833° or 0.5826 radians) is measured from the equator. Since φ is measured from the North Pole, then $\varphi_0 = \frac{\pi}{2} - 0.5826 = 0.9881$ radians. Thus

$$\omega = \cos(0.9881) = 0.55 \text{ radians per unit time.}$$

Since the duration of one complete rotation is 2π units of time with our parameterization in t, the Foucault pendulum will complete $0.55(2\pi)$ radians of rotation or 0.55 revolutions in one day.

Had the pendulum in PSF been operational, your observations would have verified this