MAT 310
Homework 11

In Problems 1-3, you will repeat the analysis you did for the relationships between area, defect, and holonomy of a hyperbolic triangle for triangles on a sphere.

1. A *spherical lune* is any of the four regions created by two different great circles. The two sides of each interior angle of a spherical triangle determine two congruent lunes with lune angle the same as the interior angle. Create the illustration of a spherical triangle and associated lunes as in Figure 1b on some spherical object (e.g., an orange or a tennis ball).

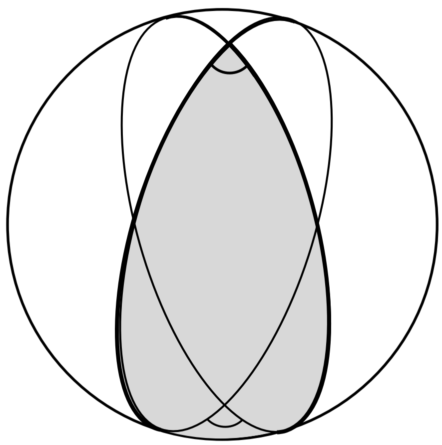


Figure 1a. Spherical lune

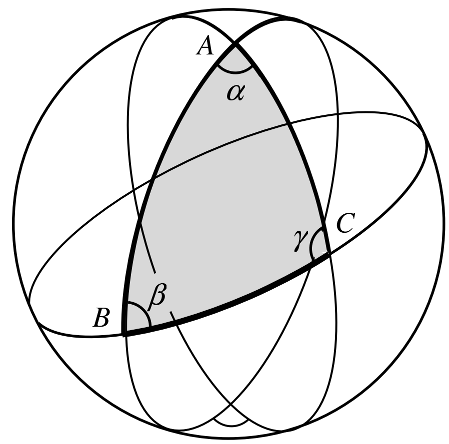


Figure 1b. Spherical triangle

   a. Describe how the three pairs of lunes determined by the three interior angles α, β, and γ, cover the sphere with some overlap. What is the overlap?
   b. Find a formula for the area of a lune with lune angle θ in terms of θ and the surface area of the sphere (of radius ρ), which you can call S_ρ. Use radian measure for angles.
   c. Find a formula for the area of a triangle on a sphere of radius ρ.

2. What can you say about the sum of the interior angles of triangles on spheres? Are there maximum and/or minimum values for the sum? Find a formula for the holonomy $\mathcal{H}(\triangle)$ of a triangle $\triangle$ on a sphere (use a triangle that lies entirely in one hemisphere).

Note: The quantity ρ^{-2} is called the *curvature* of the sphere and $-\rho^{-2}$ is called the *curvature* of the hyperbolic plane. In the following problems, use K to represent the curvature.

3. Draw an irregular octagon and dissect it into triangles without adding extra vertices. Refer to this diagram to argue that if that Γ is any simple polygon in an open hemisphere or in a hyperbolic plane, then

$$\mathcal{H}(\Gamma) = [\,\Sigma\alpha_i - (n-2)\,\pi\,] = K\ \text{area}(\Gamma)$$

   where $\Sigma\alpha_i$ is the sum of the interior angles, n is the number of sides, and K is the curvature.

4. Suppose that a geometric space can be decomposed into f simple polygons (faces) with curvature K by e line segments (edges) in the space. Suppose that these edges meet in v endpoints (vertices). Apply the result of Problem 3 to each face of the space to argue that

$$K\ \text{area}(M) = 2\pi\,(\,v - e + f\,).$$

The quantity $\chi = v - e + f$ is called the Euler characteristic of the space.

Hints: Use the notation

$\Sigma\alpha_i$ (1^{st} face), $\Sigma\alpha_i$ (2^{nd} face), …, $\Sigma\alpha_i$ ($f^{\,th}$ face) to represent the sum of the interior angles of the faces,

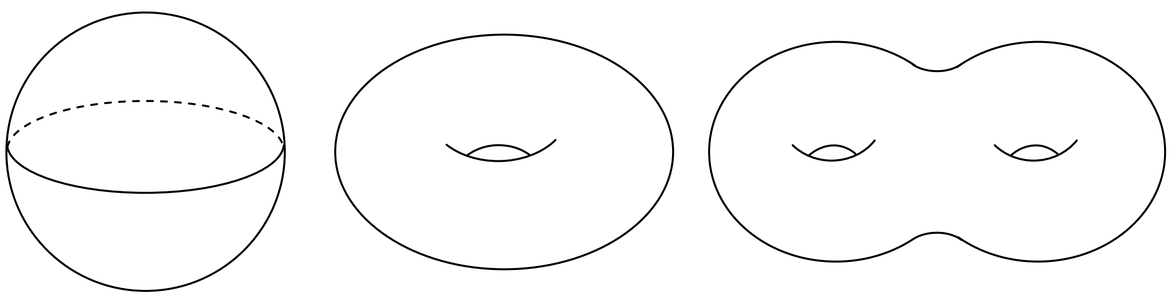
and $n_1, n_2, \ldots, n_f$ to represent the number of sides for each face.

Also, consider the following two questions:

What is the sum of the angles meeting at any one vertex?

How many faces touch each edge?

5. Draw cell divisions for a sphere, a torus, and a two-holed torus.



a. Compute the Euler characteristic for each figure.
b. Redraw each figure distorted so that it has only flat faces. In order to do this, you will have to "squeeze all of the curvature" into some number of cone points. Compute the holonomy around each cone point then find the sum of the holonomies for each figure.
c. Draw a continuous vector field with isolated zeros on each of the spaces. Then draw each zero on a separate graph with a path around it showing how the vectors point at several locations along the path. Compute the index of each zero and find the sum of the indexes of all zeros. How are these results related to the Euler characteristic?