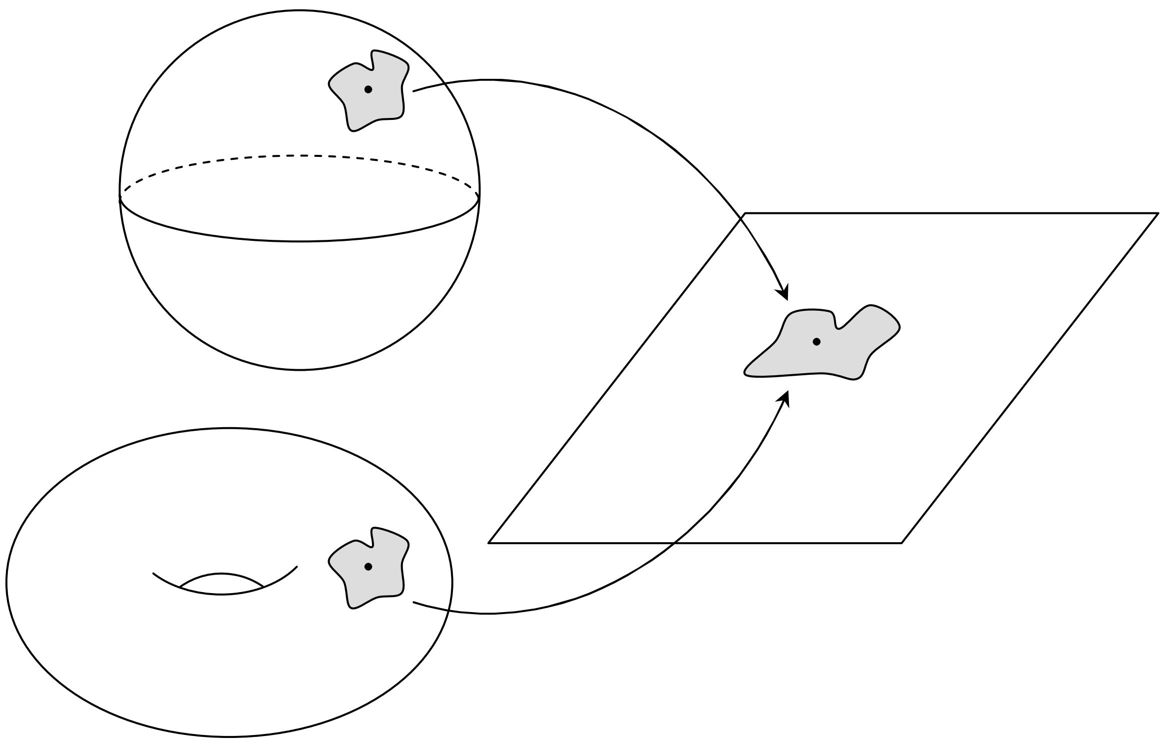
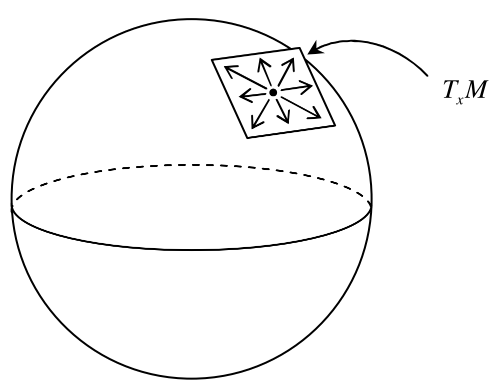
Differential Geometry Handout 1

We refer to any space that locally looks like pieces of $\mathbf{R}^n$, a *manifold*. Since it is easiest to visualize, we will focus our discussion on two-dimensional manifold such as a plane, sphere, the Poincaré disk, a torus, etc. If you look at only a small neighborhood in any of these spaces, you would see something that just looks like a portion of $\mathbf{R}^2$.



We are particularly interested in vectors that are tangent to these surfaces at various points. We will measure angles as angles between vectors. We will measure distance along a path by integrating velocity vectors along the path. Specifically we define:

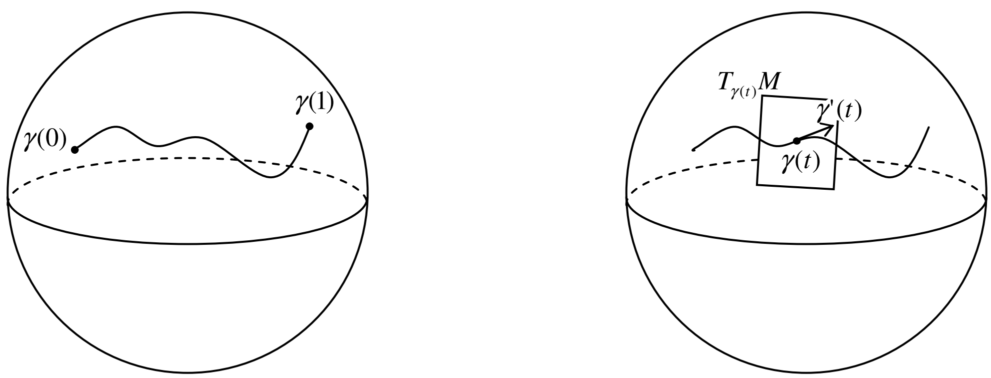
The *tangent plane* of a manifold *M* at a point *x* is the set of all vectors tangent to *M* based at *x*. We denote this tangent plane as $T_x M$.



Intrinsic and Extrinsic

In general, mathematical objects within manifold itself (such as points or paths) or vectors in the tangent planes of the manifold can be described purely in terms of the space itself. That is, reference to the space in which the manifold is embedded (the "ambient space") are not necessary. We call such objects and properties that can be derived *without* reference to the ambient space *intrinsic.*

For example, a differentiable function that maps the unit interval into the manifold $\gamma:[0,1]\to M$ describes a path in the manifold. Each number *t* in the interval [0,1] is assigned a point in the manifold. We will often think of *t* as time, in which case the path describes movement from the point $\gamma(0)$ to $\gamma(1)$. Such a path is intrinsic since we did not need to use anything about the ambient space. Likewise, if we differentiate this function, then the velocity vector $\gamma'(t)$ will be tangent to the curve at $x=\gamma(t)$. Since it is tangent to the curve and the curve lies completely in *M*, the velocity vector $\gamma'(t)$ will be in the tangent plane T_xM. Thus the first derivative of a path (the velocity vectors) are also intrinsic.

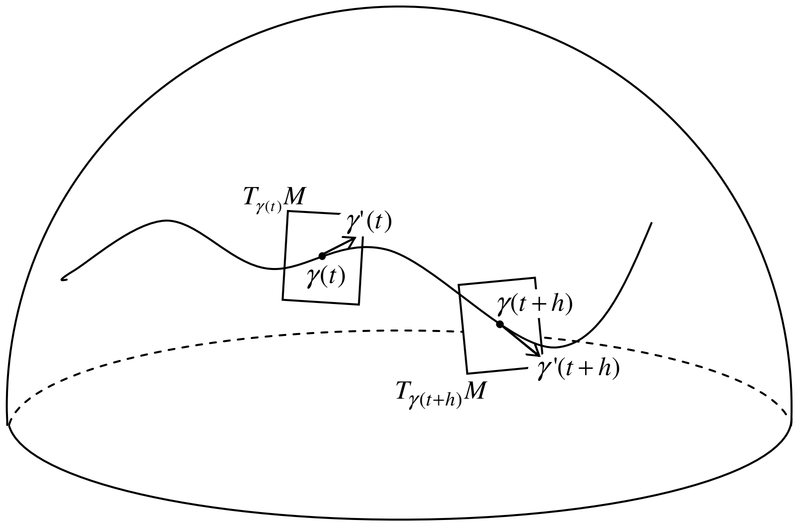


On the other hand, some important vectors do point out of the tangent plane. Such objects and properties derived from them are generally called *extrinsic*. Describing them requires the use of the ambient space in which the manifold is embedded. For example, the acceleration or second derivative of a path at some point is not necessarily a vector in the tangent plane at that point. Acceleration is extrinsic to the manifold.

Another way to see the problem that the concept of acceleration presents for intrinsic geometry is to think about the definition of the derivative,

$$\text{acceleration} = \gamma''(t) = \tfrac{d}{dt}\big(\gamma'(t)\big) = \lim_{h\to 0}\frac{\gamma'(t+h)-\gamma'(t)}{h}.$$

The problem is that if $\gamma(t)$ and $\gamma(t+h)$ are different points (which is likely), then the vectors $\gamma'(t)$ and $\gamma'(t+h)$ are vectors in different tangent planes. Within the intrinsic geometry of our manifold, we have no way of comparing these two vectors. Specifically, there is no meaningful way to subtract them as suggested by the difference quotient. We would have to resort to referring to the ambient space to subtract these vectors making the second derivative an extrinsic object.



More generally than acceleration, we will also want to consider the rate at which any set of vectors changes along some path. If **V** is a vector field on the manifold *M* then at each point $x \in M$, we will get a vector $\mathbf{V}(x) \in T_x M$. Then given a path $\gamma : [0,1] \to M$, we can use the ambient space to define the derivative of **V** at any point $\gamma(t)$ on the path with the usual difference quotient

$$\text{rate} = \tfrac{d}{dt}\left(\mathbf{V}(\gamma(t))\right) = \lim_{h \to 0} \frac{\mathbf{V}(\gamma(t+h)) - \mathbf{V}(\gamma(t))}{h}.$$

We encounter the same problem that we had with acceleration: if $\gamma(t)$ and $\gamma(t+h)$ are different points, then the vectors $\mathbf{V}(\gamma(t))$ and $\mathbf{V}(\gamma(t+h))$ are in different tangent planes. Within the intrinsic geometry of our manifold, we cannot subtract them. Thus, this derivative is an extrinsic object.

The No-Turning Condition and the Need for an Intrinsic Derivative.

Recall that we defined parallel transport along a "straight line" in a space by using a "no turning" condition. Specifically, if a "straight line" is by definition a path that "doesn't turn," then keeping the angle between the path and a vector constant means that the vector is also not turning. Thus we have a need to talk about no-turning for a curve. Unfortunately, the turning of a curve must be defined in terms of the rate of change of the velocity vector (in other words, the acceleration). Thus we need a meaningful intrinsic definition of acceleration. More generally, in order to define parallel transport along paths that aren't straight, we will need to define a derivative for vectors along a path that is also intrinsic.

It turns out that there are many ways to take such a derivative, and they are all called *connections* because they require some way of "connecting" tangent spaces at different points in order to determine a rate of change. However, there is only one way to take such a derivative that also keeps the geometry intact (technically, conditions referred to as "torsion-free" and "metric-compatible"). This unique way is equivalent to taking the derivative in the ambient space (in our case of dealing with surfaces, this will usually just be $\mathbf{R}^3$) then project the result onto the tangent plane.

For acceleration, you can think of this intrinsic derivative as breaking the acceleration vector into two components. One component, perpendicular to the tangent plane, is the acceleration required to keep us moving in our space, that is keep the path on the manifold. This portion of the acceleration has purely to do with how the manifold is embedded in the ambient space and is the extrinsic part of the acceleration. The other component of acceleration will lie in a tangent plane of the manifold. This portion of the acceleration determines whether motion along the path is speeding up or turning in the space, and it is the intrinsic derivative.

Thus we define the *intrinsic derivative* (also called the *covariant derivative* or *Levi-Civita Connection*) of a vector field at a point x in the direction of a vector $\mathbf{W} \in T_x M$ to be:

$$\nabla_{\mathbf{W}} \mathbf{V} = \text{Proj}_{T_x M} \tfrac{d}{dt}\left(\mathbf{V}(\gamma(t))\right),$$

where the curve γ is defined so that $x = \gamma(t)$ and $\mathbf{W} = \gamma'(t)$,

$\text{Proj}_P\, \mathbf{v}$ is the projection of vector $\mathbf{v}$ onto plane P,

and $\frac{d}{dt}$ is the standard differentiation operator in $\mathbf{R}^3$.

Differentiating the velocity vector along a curve γ results in the intrinsic acceleration,

$$\nabla_{\gamma'(t)} \gamma'(t) = \text{Proj}_{T_{\gamma(t)} M}\, \gamma''(t).$$

Now what would a no-turning condition look like? If the intrinsic acceleration vector $\nabla_{\gamma'(t)} \gamma'(t)$ is pointing in the same direction (or opposite direction) as the velocity vector $\gamma'(t)$, then the motion on the path is simply speeding up (or slowing down). It would not be turning. Thus we can define a *straight line* (also called a *geodesic*) to be a curve $\gamma : [0,1] \to M$ with the property that

$$\nabla_{\gamma'(t)} \gamma'(t) = c\gamma'(t) \text{ for some } c \in \mathbf{R}.$$

If we are moving at a constant speed on a geodesic, this condition becomes

$$\nabla_{\gamma'(t)} \gamma'(t) = 0.$$

Likewise we can now define parallel transport along paths that are not geodesics with a similar condition:

$$\nabla_{\gamma'(t)} \mathbf{V}\left(\gamma(t)\right) = 0.$$