MAT 294 Exam 1
Fall 2005

Name (printed neatly): ___________________________

Each of the six numbered questions on this exam is worth 25 points.

1. Show all of your work in finding the limit $\lim_{x\to 1}\dfrac{x^2+x-2}{x-1}$.

   (Hint: The final answer is 3.)

$$\lim_{x\to 1}\frac{x^2+x-2}{x-1} = \lim_{x\to 1}\frac{(x+2)(x-1)}{x-1} = \lim_{x\to 1} x+2 = 3$$

2. Use the limit definition to compute the derivative of $f(x)=\sqrt{2x+1}$ at $x=2$. Show all of your work.

   (Hint: The final answer is $f'(2)=\dfrac{1}{\sqrt{5}}$.)

$$f'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{\sqrt{2(2+h)+1}-\sqrt{5}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{2h+5}-\sqrt{5}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{2h+5}-\sqrt{5}}{h}\cdot\frac{\sqrt{2h+5}+\sqrt{5}}{\sqrt{2h+5}+\sqrt{5}}$$

$$= \lim_{h\to 0}\frac{(2h+5)-5}{h\left(\sqrt{2h+5}+\sqrt{5}\right)} = \lim_{h\to 0}\frac{2}{\sqrt{2h+5}+\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}}$$

3. In Question #2, you showed that $f'(2) = \frac{1}{\sqrt{5}}$ for the function $f(x) = \sqrt{2x+1}$. Use this result to find the equation of the tangent line to the graph of *f* at $x = 2$. Show all of your work.

$f(2) = \sqrt{5}$ slope $= \frac{1}{\sqrt{5}}$ point $= (2, \sqrt{5})$

$y - \sqrt{5} = \frac{1}{\sqrt{5}}(x-2)$

$y = \frac{1}{\sqrt{5}}(x-2) + \sqrt{5}$

4. In Question #2, you showed that $f'(2) = \frac{1}{\sqrt{5}}$ for the function $f(x) = \sqrt{2x+1}$. Using $x = 2$ and $\Delta x = dx = \frac{1}{2}$, find values for the following quantities. Also explain briefly what they mean in terms of the graph of *f*. Show all of your work.

Δy	Change in y on the graph of f. $f(2.5) = \sqrt{6}$ so $\Delta y = \sqrt{6} - \sqrt{5}$ $f(2) = \sqrt{5}$
dy	change in y on the tangent line. Slope $= \frac{1}{\sqrt{5}}$ $dy = \frac{1}{\sqrt{5}} \cdot \frac{1}{2} = \frac{1}{2\sqrt{5}}$ $dx = 1/2$
$\frac{\Delta y}{\Delta x}$	Slope of the secant line $\Delta y = \sqrt{6} - \sqrt{5}$ $\frac{\Delta y}{\Delta x} = 2(\sqrt{6} - \sqrt{5})$ $\Delta x = \frac{1}{2}$
$\frac{dy}{dx}$	slope of the tangent line $\frac{dy}{dx} = f'(x) = \frac{1}{\sqrt{5}}$

5. An object is falling according to the equation $h(t) = 100 - 16t^2$ feet (with t measured in seconds).

   a. Using the approximation techniques discussed in class, find an ***under-estimate*** for the speed of the object when $t = 2$ seconds. You do not need to simplify your answer using a calculator, but may do so if you wish.

$$\text{Speed} = h'(2) \approx \frac{h(2) - h(1.9)}{2 - 1.9} = \frac{(100 - 16\cdot 2^2) - (100 - 16\cdot 1.9^2)}{0.1}$$

$$= \frac{36 - 42.24}{0.1} = -62.4 \text{ ft/s}$$

   b. Explain carefully how you know this is an under-estimate.

The object is speeding up, so in the time interval from $t = 1.9$ to $t = 2$, it is traveling slower than the speed at $t=2$. So the average speed during this interval will be an under-estimate for the instantaneous speed at the end of the interval.

6. An object is falling according to the equation $h(t) = 100 - 16t^2$ feet (with t measured in seconds). You can approximate the speed at $t = 2$ seconds as in the previous question and make it more accurate by making Δt smaller.

   Note: Although you have the tools at this point to determine that the speed is exactly 64 ft/s at $t = 2$ seconds, you may not use that information in this problem.

   a. Determine how small you would need to make Δt in order to make the error less than $\varepsilon = 0.1$ ft/s. (Hint: It may be helpful to simplify your expressions for under and overestimates as much as possible.)

$$\text{underestimate} = \frac{h(2) - h(2-\Delta t)}{\Delta t} = \frac{36 - [100 - 16(2-\Delta t)^2]}{\Delta t}$$

$$= \frac{36 - 36 - 64\Delta t + 16\Delta t^2}{\Delta t} = 16\Delta t - 64$$

$$\text{overestimate} = \frac{h(2+\Delta t) - h(2)}{\Delta t} = \frac{[100 - 16(2+\Delta t)^2] - 36}{\Delta t}$$

$$= \frac{36 - 64\Delta t - 16\Delta t^2 - 36}{\Delta t} = -16\Delta t - 64$$

$$\text{bound} = |\text{overestimate} - \text{underestimate}| = |-16\Delta t - 64 - 16\Delta t + 64|$$

$$= 32\Delta t < 0.1$$

$$\Delta t < 0.1/32$$

   b. Find a general formula to determine how small you would need to make Δt in order to make the error less than ε ft/s. Here ε must be treated as a variable so that you could plug in any value and get an appropriate Δt.

$$\text{bound} = 32\Delta t < \varepsilon$$

$$\Delta t < \varepsilon/32$$