MAT 294 Written Homework 3
Fall 2005 Solutions

1a. The average rate of change over the first hour of eating is

$$\frac{F(1)-F(0)}{1-0}=\frac{\left(-10.28+175.9\cdot 1\cdot e^{-1/1.3}\right)-\left(-10.28+175.9\cdot 0\cdot e^{-0/1.3}\right)}{1}$$
$$=175.9e^{-1/1.3}$$
$$=81.5 \text{ kJ/hr/hr}$$

In terms of the thermic effect of food, this means that the metabolic rate increases b by 81.5 kJ/hr over the course of the first hour.

1b. Using the MATLAB program, with

```
f=-10.28+175.9*x*exp(-x/1.3);
a=1;
epsilon=.01;
```

we get a result of 18.8 kJ/hr^2. This means that one hour after eating the instantaneous rate of change of the metabolic rate is 18.8 kJ/ hr^2. In other words, if the metabolic rate were to continue changing at this constant rate (following the linear model, or tangent line, from that point) it would increase by 18.8 kJ/hr every hour.

1c. We are approximating the instantaneous rate of change of the metabolic rate with respect to time, one hour after eating. This is represented algebraically by the derivative

$$F'(1)=\lim_{h\to 0}\frac{F(1+h)-F(1)}{h}=18.8 \text{ kJ/hr}^2.$$

Graphically, this is the slope of the tangent line.

The approximations are average rates of change of the metabolic rate over small time periods of duration h, either starting or ending at t=1, which is the difference quotient from the limit above. Graphically, these are the slopes of secant lines. Using h=0.01 for example yields the approximations

22.85 kJ/hr^2 and 15.13 kJ/hr^2

The errors are the differences between the desired instantaneous rate and the approximating average rates:

$$\text{error} = \left|\frac{F(1+h)-F(1)}{h}-F'(1)\right|$$

Using the approximations from above, we have

error #1 = | 22.85 – F ’(1) |
error #2 = | 15.13 – F ’(1) |

Graphically the errors are the differences between the slope of the tangent line and the slopes of the approximating secant lines.

The metabolic rate is increasing at a decreasing rate for the first 1.3 hours, that is, the graph is concave down. This means that average rates to the left of t=1 will be overestimates and average rates to the right of t=1 will be underestimates. Thus we can bound the error by taking the difference between an overestimate and an underestimate. Specifically,

$$\text{error} < \frac{F(1-h)-F(1)}{-h} - \frac{F(1+h)-F(1)}{h}$$

where h is a small positive number. Using the approximations from above, we get that the error < 22.85 – 15.13 = 7.72. Graphically, this bound for the error is the difference between the slopes of the secant lines taken on either side of the point at time t=1.

We can make the errors as small as we want by making $h \to 0$. Specifically, we can find under and overestimates and determine the bound (the difference between the two). We can keep taking smaller values of h until the bound is as small as we need it to be. By running the MATLAB program, we can set epsilon=.01 and get an overestimate of 18.8131 kJ/hr^2 and an underestimate of 18.8054 kJ/hr^2. Graphically, we are making the slopes of the secant lines very close to each other, and the slope of the tangent must be between these values.

2a. The average rate of change from t=200 seconds to t=201 seconds is

$$\frac{Q(201)-Q(200)}{201-200} = 10^{-5}\cdot 10\cdot\left(1-e^{-201/10^7\cdot 10^{-5}}\right) - 10^{-5}\cdot 10\cdot\left(1-e^{-200/10^7\cdot 10^{-5}}\right)$$

$$= 1.3466\times 10^{-7} \text{ C/s.}$$

2b. Since the rate is about 10^{-7} amps (A), we need to be within epsilon= 5×10^{-10} of the actual value. Using the MATLAB program with

```
f=1e-5*10*(1-exp(-x/(1e7*1e-5)));
a=200;
epsilon=5e-10;
```

we get a charging current (rate) of 1.35×10^{-7} A.

2c. We are approximating the current in the circuit 200 seconds after the switch has been closed. This is the same thing as the instantaneous rate of change of the charge with respect to time. Algebraically, this can be represented as

$$Q'(200) = \lim_{h\to 0}\frac{Q(200+h)-Q(200)}{h}$$

$$= \lim_{h\to 0}\frac{CV\left(1-e^{-(200+h)/RC}\right) - CV\left(1-e^{-200/RC}\right)}{h}$$

Graphically, this is the slope of the tangent line to the graph of $Q(t)$ at t=200.

The approximations are the average currents (i.e., average rates of change of the charge) over small time intervals starting or ending at t=200 seconds. Symbolically these are the difference quotients

$$\frac{CV\left(1-e^{-(200+h)/RC}\right) - CV\left(1-e^{-200/RC}\right)}{h}$$

And using the values of C, V, and R given in the problem and a time interval of one second, we get approximations of

$$1.3601\times10^{-7}\text{ A} \quad \text{and} \quad 1.3466\times10^{-7}\text{ A}.$$

Graphically, these are the slopes of secant lines with $\Delta t = 1$ starting and ending at t=200.

The errors are the differences between these average currents and the actual current at t=200 s. That is,

$$\text{error} = \left|\frac{CV\left(1-e^{-(200+h)/RC}\right)-CV\left(1-e^{-200/RC}\right)}{h}-Q'(200)\right|$$

and for the approximations above, the errors are

$$\text{error \#1} = \left|1.3601\times10^{-7}-Q'(200)\right|$$

$$\text{error \#2} = \left|1.3466\times10^{-7}-Q'(200)\right|.$$

Graphically, the errors are the differences between the slope of the tangent line to the graph of $Q(t)$ at t=200 and the slopes of the secant lines.

Since the current is decreasing (i.e., the charge is increasing at a decreasing rate or in other words, the graph of $Q(t)$ is concave down), we will get overestimates for time periods before t=200 s and underestimates for time periods after t=200 s. Thus we can use the differences between an under and overestimate to bound the error. We can express this generally as

error < overestimate - underestimate

$$=\frac{CV\left(1-e^{-(200-h)/RC}\right)-CV\left(1-e^{-200/RC}\right)}{-h}-\frac{CV\left(1-e^{-(200+h)/RC}\right)-CV\left(1-e^{-200/RC}\right)}{h}$$

for small positive values of h. For the approximations above, we get

$$\text{error} < 1.3601\times10^{-7}-1.3466\times10^{-7} = 0.0135 \text{ amps.}$$

Graphically, this is the difference between the slopes of the secant lines taken to the left and right of the point at t=200.

We can make the errors as small as we want by making $h \to 0$. Specifically, we can find under and overestimates and determine the bound (the difference between the two). We can keep taking smaller values of h until the bound is as small as we need it to be. By running the MATLAB program, we can set epsilon=5e-10 and get an overestimate of 1.3540×10^{-7} amps and an underestimate of 1.3527×10^{-7} amps. Graphically, we are making the slopes of the secant lines very close to each other, and the slope of the tangent must be between these values.