WORKSHEET 17

1. a) Give the conditions necessary for a function $f\colon \mathbb{R} \to \mathbb{R}$ to have a Taylor series expansion about a point $c \in \mathbb{R}$.

   b) Graph the function $f(x) = \ln x$. What is the domain of f?

   c) For what values of $c \in \mathbb{R}$ does f has a Taylor series centered at c? Compute the Taylor series for two of these values c_1 and c_2.

   d) What is the interval of convergence for these two Taylor series?

   e) Write out $P_5(x; c_1)$ and $P_5(x; c_2)$, the 5^{th}-order Taylor polynomials for $f(x) = \ln x$ centered about c_1 and c_2 respectively. Choose some real number x_0 that is inside the interval of convergence for both Taylor series, and compute $P_5(x_0; c_1)$ and $P_5(x_0; c_2)$.

   f) From Lagrange's Theorem, what can be said about the error involved in using these two Taylor polynomials to estimate $\ln x_0$.
   (Hint: For each expansion, what is the maximum value of $f^{(6)}(x)$ on the interval from x_0 to the center of the expansion?)

   g) Use your calculator to find $\ln x_0$. How does your calculator come up with this number?

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function. (What does that mean?) Only some of the following statements are true. For those that are true, indicate why. For those that are false, provide a counterexample.

   a) If a Taylor series expansion for f converges at $x \in \mathbb{R}$, then it converges to $f(x)$.

   b) If the remainder $R_n(x; c)$ tends to zero as $n \to \infty$, then the Taylor series converges at x.

   c) If the remainder $R_n(x; c)$ tends to zero as $n \to \infty$, then the Taylor series converges at x, and it converges to $f(x)$.

   d) If a Taylor series converges to $f(x)$ for some values of x, then it converges to $f(x)$ for all values of x in the interval of convergence.

   e) If all the derivatives of f are bounded on some interval containing the center of expansion, then the Taylor series converges to f on this interval.

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3. Taylor series may be found for each of the following functions by a long method or a short method. Describe both methods, then use the short one. Determine the interval of convergence in each case.

   a) $f(x) = x^3 e^x$ b) $f(x) = \dfrac{1}{1 - x^2}$ c) $f(x) = \dfrac{1}{1 + x^2}$

   d) $f(x) = \ln(x^2 + 1)$ e) $f(x) = \arctan x$

4. A flock (or is it gaggle?) of geese in flight will always travel in a "V-shaped" formation. One side of this "V", however, is almost always longer than the other. Using rigorous mathematics, explain why this happens.