WORKSHEET 4

1. The definite integral has many applications only one of which is the problem of finding the area under a curve. State explicitly what a definite integral is. Now give some examples of problems to which this applies.

2. From physics recall that the force acting on an object is equal to its mass times its acceleration, and that work is equal to the force acting on an object multiplied by the distance the object has traveled ($F = ma$ and $W = Fd$.) In lifting a mass, the force to be overcome is that of gravity which (on the surface of the Earth) corresponds to an acceleration of 9.8 m/s^2.

   a) Suppose a 1 kg bucket filled with 10 kg of water (equals approximately 10 liters - isn't the metric system great?) is lifted 10 m into the air.

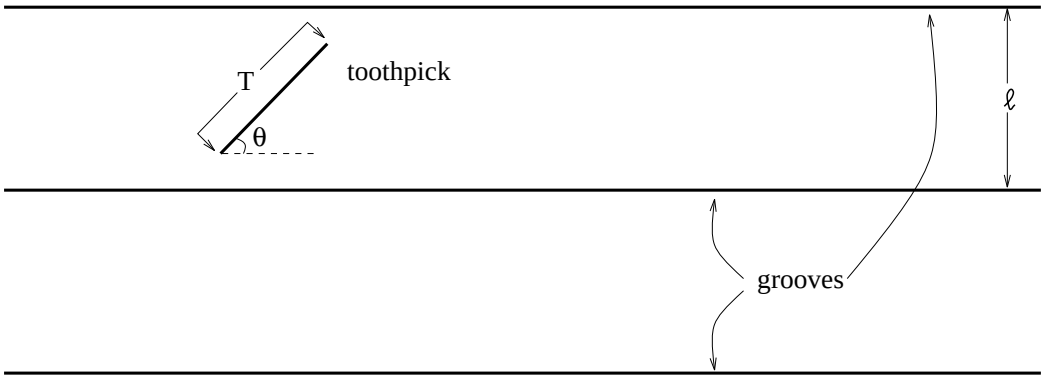
   Set up an integral which gives the total amount of work done on the bucket and the water in lifting it. Evaluate the integral.

   b) Now suppose a bucket of water is lifted 10 meters into the air in such a way that (i) there are 10 kg of water in the bucket at the start, (ii) the bucket is lifted at a constant rate, (iii) the water leaks out of the bucket at a constant rate, (iv) the bucket contains only 5 kg (how many liters?) of water when its is 10 m high.

   Set up an integral which gives the total amount of work done on the bucket and the water in lifting it. Evaluate the integral.

   How much work is required to lift the bucket 20 m in this situation? 30 m?

3. Imagine a hardwood floor which has wooden panels 5 cm wide arranged parallel along the floor. The grooves between adjacent panels form parallel lines 5 cm apart. If you drop a 5 cm long toothpick on the floor, what is the chance that it will straddle one of these grooves?



   a) When the tothpick lands, it must form some angle relative to the parallel grooves. Call this angle θ. For a fixed value of θ, let $P(\theta)$ be the probability that the toothpick straddles a groove. Find a formula for $P(\theta)$.

   b) Now determine the average value of $P(\theta)$. (over what interval?) Since each angle is equally likely, this should give you the probability that the toothpick straddles a groove.

Mike Oehrtman

c) Now suppose the grooves are spaced at a distance of ℓ from each other and the length of the toothpick is T. Now what is the probability that the toothpick straddles a groove?

d) Try it out. Measure a pencil or something else and use the grooves between tiles on the floor (only in one of the directions). Toss your pencil 10 or 20 times and see how many times it straddles a groove in the direction you chose. How close was this to the theoretical value? If the values are not very close, see if you can explain why.

4. In problem 2, the acceleration due to gravity was given as the constant 9.8 m/s^2. In actuality, the force of gravity that the earth exerts on an object diminishes as the object gets further away from the earth. The work required to lift an object 1 foot at sea level is greater than the work required to lift the same object the same distance at the top of Mt. Everest. However, the difference in altitudes is so small in comparison to the radius of the earth that the difference in work is negligible. On the other hand, when an object is rocketed into space, the fact that the force of gravity diminishes with distance from the center of the earth is critical.

   According to Newton, the force of gravity on a given mass is proportional to the reciprocal of the square of the distance of that mass from the center of the earth. That is, there is a constant k such that the gravitational force at distance r from the center of the earth, $F(r)$, is given by

$$F(r) = \frac{k}{r^2}\,.$$

   How much work (in mile-pounds) is required to lift a 1-pound payload from the surface of the earth to the moon, which is about 240,000 miles away? (The earth's surface is at a distance of 4,000 miles from its center.)

   (Hint: When $r = 4,000$ (miles) the force is 1 pound. Use this to determine the constant k.)

5. The kinetic energy of an object with mass m and speed v is $\frac{1}{2}mv^2$, at least in the case where the entire object is moving at the same speed. Suppose a rod with length 10 cm and density 3 grams per centimeter is rotating around one of its ends at a rate of one revolution per minute, much like the second hand of a clock.

   a) Find the kinetic energy of the rod.

   b) If the rod is only half as long but moves twice as fast, does the kinetic energy increase or decrease?

6. Suppose that you have two magnets and a wire. One magnet is attached to the end of the wire and the other can slide along the wire. If the magnets are arranged so that they repel each other, then it will require force to move the movable magnet toward the fixed magnet. The amount of force needed to move the magnet increases as the two get closer together. In fact, the force at a distance d is proportional to $1/d^2$. Which will require more work, to move the magnet from 5 cm away to 3 cm away, or from 3 cm away to 2 cm away?