John Smith
MAT 266
Fall 2010

Fourier Series Approximation of Periodic Square Waves

A Fourier Series (or trigonometric series) can be very useful (and in some cases better suited than power series) for expressing periodic events such as astronomical phenomena heartbeats, tides, and vibrating strings as mathematical functions. To the human ear, musical tones separated by a factor of 2 in frequency (an octave) sound very similar to each other. In Western music, an octave is divided into 12 notes that are equally spaced on a logarithmic scale. The harmonics of a sound wave are integer multiples of the fundamental frequency (F_0 defined as the lowest frequency of a periodic sound wave) and are the component frequencies of the signal. Since harmonics all have the property of being periodic at the fundamental frequency, the sum of harmonics is also always periodic at that frequency.

A Fourier Series can be used to approximate a square wave, a type of sound wave typically associated with the distinct harmonics of digital sounds (prevalent in early arcade and computer games). To find the Fourier Series for a square wave given by the function:

$$f(x) = \begin{cases} 0, & -\pi \leq x < 0 \\ 1, & 0 \leq x < \pi \end{cases}$$

$$\text{and } f(x) = f(x + 2\pi)$$

f(x) is a periodic function and repeats itself after every 2π. Using the definition of a Fourier Series (see source), a Fourier Series approximating f(x) can be found as follows:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} 1\,dx\right) = \frac{1}{2\pi}(\pi - 0) = \frac{1}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}\cos nx\,dx\right) = \frac{1}{\pi}\left(0 + \frac{\sin nx}{n}\right)\Big|_0^{\pi}$$

$$= \frac{1}{n\pi}(\sin nx - \sin 0) = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}\sin nx\,dx\right) = \frac{1}{\pi}\left(0 - \frac{\cos nx}{n}\right)\Big|_0^{\pi}$$

$$= -\frac{1}{n\pi}(\cos n\pi - 1)$$

$$= \begin{cases} 0, & if\ n\ is\ even \\ \dfrac{2}{n\pi}, & if\ n\ is\ odd \end{cases}$$

Therefore the Fourier Series approximation for f is:

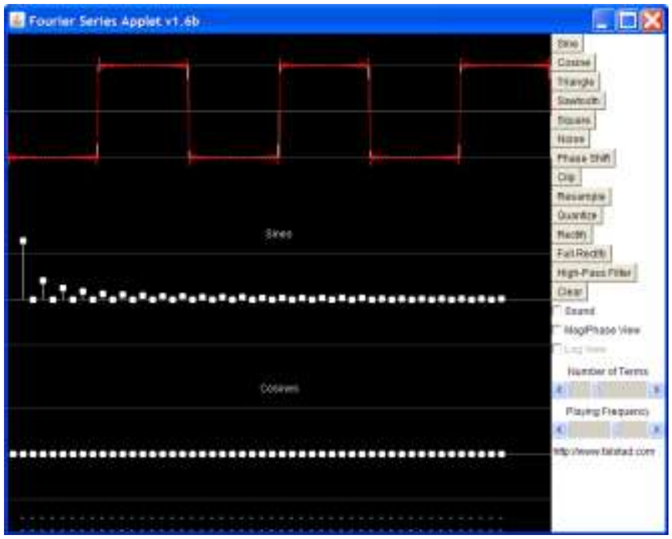
$$a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots$$

$$= \frac{1}{2} + 0 + 0 + 0 + \cdots + \frac{2}{\pi}\sin x + 0\sin 2x + \frac{2}{3\pi}\sin 3x + \cdots$$

By setting $n = 2k - 1$ (odd integers) and using sigma notation, the approximation of f can be rewritten using sigma notation as:

$$\frac{1}{2} + \sum_{k=1}^{\infty}\frac{2}{(2k-1)\pi}\sin((2k-1)x)$$

Approximating a square wave using a Fourier Series can be useful for running simulations, building computer programs etc., such as in the java applet below.



(Screenshot from java applet found at http://www.falstad.com/fourier/)

References

Johns Hopkins University. (n.d.). *Fourier Series*. Retrieved November 30, 2010, from http://www.jhu.edu/signals/listen-new/listen-newindex.htm

Stewart Calculus. (n.d.). *Fourier Series*. Retrieved November 30, 2010, from http://www.stewartcalculus.com/data/CALCULUS%20Early%20Transcendentals/upfiles/FourierSeries5ET.pdf

Wikipedia. (2010, November 24). *Fundamental frequency*. Retrieved November 30, 2010, from http://en.wikipedia.org/wiki/Fundamental_frequency

Wikipedia. (2010, November 25). *Harmonic*. Retrieved November 30, 2010, from http://en.wikipedia.org/wiki/Harmonic