Extreme Values: Boundaries and the Extreme Value Theorem

In our discussion of maxima and minima of functions of a single variable in Section 12.1, we saw that extrema frequently occurred at endpoints of the domain. To generalize this idea to functions of more than one variable, we should think of the endpoints of the domain of a function of a single variable as *boundary points* of the domain; for instance, the boundary of the closed interval [1, 3] consists of the two endpoints 1 and 3, whereas the open interval (1, 3) has no boundary points. (The boundary points 1 and 3 are outside the interval.)

As we have seen, the domains of functions of two variables are subsets of the plane; for instance, the natural domain of the function $f(x, y) = \sqrt{x^2 + y^2 - 1}$ consists of all points (x, y) in the plane with $x^2 + y^2 - 1 \geq 0$, or $x^2 + y^2 \geq 1$, and its boundary is the unit circle (Figure 1).

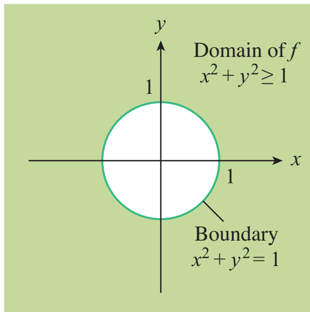


Figure 1

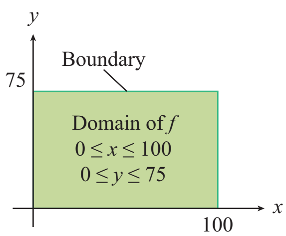


Figure 2

Even if the function is mathematically defined for all x and y, we may need to restrict the domain in real life applications: For example, if $C(x, y) = 10{,}000 + 20x + 40y$ is the monthly cost of producing x Ultra Mini speakers and y Big Stack speakers (see Example 1 of Section 15.1), then the function makes sense only if $x \geq 0$ and $y \geq 0$, and would be restricted further by the maximum number of Ultra Minis and Big Stacks the company can produce in a month—for instance, $x \leq 100$ and $y \leq 75$. The domain of the cost function would then be the region of the plane described by $0 \leq x \leq 100$ and $0 \leq y \leq 75$ (Figure 2), and its boundary is a rectangle.

We can now state the counterpart of the method we used in Chapter 12 to locate maxima and minima:

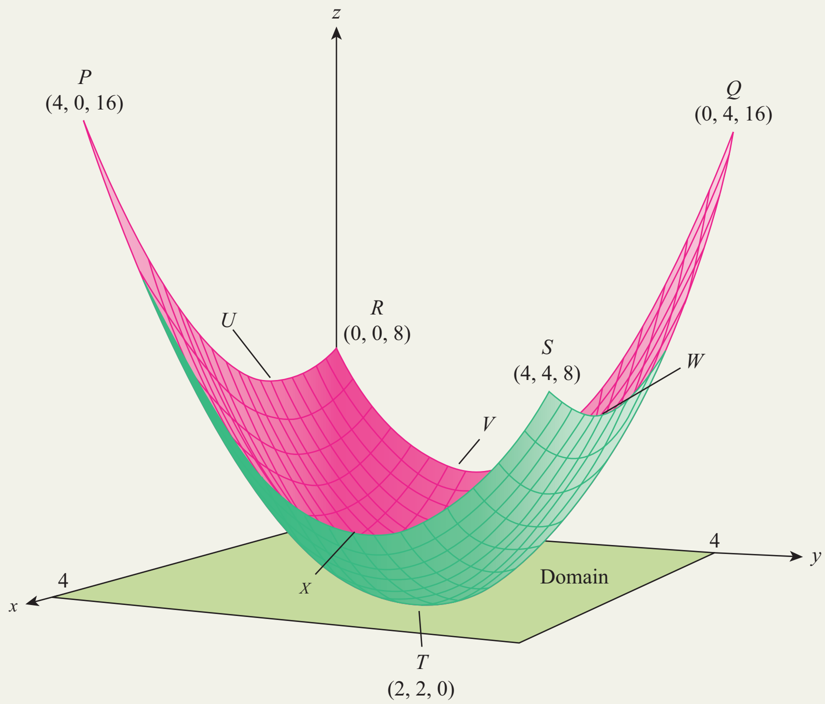
Locating Candidates for Extrema for a Function *f* of Two Variables

Step 1: ***Locate critical points in the interior of the domain.***
To locate interior points, we use the method discussed in Section 15.3: Set $f_x = 0$ and $f_y = 0$ simultaneously, and solve for x and y.

Step 2: ***Locate extrema on the boundary of the domain.***
To locate critical extrema on the boundary of the domain, we examine the behavior of the function along each segment of the boundary.

Quick Example

Consider the function f with domain $0 \le x \le 4$, $0 \le y \le 4$, whose graph is shown here:



$f(x, y) = (x - 2)^2 + 2(y - 2)^2 - (x - 2)(y - 2)$
$0 \le x \le 4,\ 0 \le y \le 4$

We observe the following:

- There is an absolute minimum at the critical point $T(2, 2, 0)$. Note that $(2, 2)$ is an interior point of the domain.
- There are absolute maxima at $P(4, 0, 16)$ and $Q(0, 4, 16)$. These are not critical points but correspond to points on the boundary of the domain (endpoints of its edges).
- There are relative maxima at $R(0, 0, 8)$ and $S(4, 4, 8)$, again corresponding to points on the boundary of the domain.
- The points U, V, W, X are critical points on boundary segments, but neither maxima nor minima. (Can you see why?)

The domains illustrated in the above examples are all **closed** sets: sets that include all their boundary points. The rectangular domain in the quick example above is also **bounded**—that is, the entire domain can be enclosed in a (large enough) disc. The domain shown in Figure 1 is **unbounded**, as it cannot be enclosed in any disc, no matter how large. The following result states that, when the domain of a continuous function is both closed and bounded, we can always expect to find an absolute maximum and an absolute minimum, as in the quick example above.

Extreme Value Theorem for Functions of Two Variables

If f is a continuous function of two variables whose domain D is both closed and bounded, then there are points (x_1, y_1) and (x_2, y_2) in D such that f has an absolute minimum at (x_1, y_1) and an absolute maximum at (x_2, y_2).

Quick Examples

1. In the quick example above, we saw from its graph that the function
$$f(x, y) = (x-2)^2 + 2(y-2)^2 - (x-2)(y-2),$$
with closed and bounded domain $0 \le x \le 4$ and $0 \le y \le 4$, has an absolute minimum at (2, 2) and absolute maxima at both (4, 0) and (0, 4).
2. If the domain is not closed and bounded, the function need not have an absolute maximum or minimum: The function
$$f(x, y) = x^2 + y^2$$
with domain all of the xy-plane has an absolute minimum at (0, 0) but no absolute maximum. (Its graph is the paraboloid shown in Example 5 of Section 15.1.)

EXAMPLE 1 Absolute Maximum: Rectangular Domain

You own a company that makes two models of stereo speakers, the Ultra Mini and the Big Stack. Your monthly profit is estimated to be

$$f(x, y) = 10x + 20y - 0.5xy.$$

Here, x is the number of Ultra Minis, y is the number of Big Stacks, and f is your profit in dollars. You can produce up to 100 Ultra Minis and 75 Big Stacks in a month. Find the number of each model you should make each week in order to maximize your profit.

Solution We are asked to find the absolute maximum value of the function f. The domain of f is specified by the production limits: $0 \le x \le 100$ and $0 \le y \le 75$, and is the one shown in Figures 2 and 3. Because the domain is closed and bounded, we know from the Extreme Value Theorem that there is a point somewhere in the domain at which f is a maximum. To identify that point, we locate all candidates for extrema in the interior of the domain and on its boundary by using the procedure outlined earlier:

Step 1: Locate critical points in the interior of the domain.
The partial derivatives are

$$f_x = 10 - 0.5y \qquad f_y = 20 - 0.5x$$

Setting these equal to zero and solving for x and y gives the only critical point as (40, 20). This point is in the interior of the domain, so it is one of our candidates for extrema.

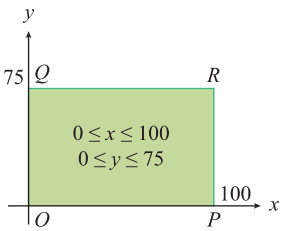


Figure 3

Step 2: Locate extrema on the boundary of the domain.
The boundary of the domain consists of four line segments *OP, OQ, QR,* and *PR* (Figure 3), which we consider one at a time.

Segment OP: $y = 0, 0 \le x \le 100$:
The behavior of the function f along this segment is seen by substituting $y = 0$ into the expression for f:

$$f(x, 0) = 10x + 20(0) - 0.5x(0) = 10x, \; 0 \le x \le 100$$

This means that, *along the line segment OP*, the value of f is determined by the function $f(x, 0) = 10x$ of a single variable, with domain $0 \le x \le 100$. We now find the relative extrema of this function of one variable by the methods we used in the chapter on applications of the derivative. The endpoints are 0 and 100, while there are no critical points because

$$f'(x, 0) = 10$$ We are taking the derivative of this function of x.

and so is never zero. The endpoints $x = 0$ and $x = 100$ give, with $y = 0$ for this segment, the points $O(0, 0)$ and $P(100, 0)$ as two candidates for extrema.

Segment OQ: $x = 0, 0 \le y \le 75$:
Substitute $x = 0$ into the function f to obtain

$$f(0, y) = 10(0) + 20y - 0.5(0)y = 20y, \; 0 \le y \le 75,$$

a function of the single variable y that determines the value of f along the segment OQ. Again, there are no critical points, only the endpoints $y = 0$ and $y = 75$. These give us $O(0, 0)$ again and $Q(0, 75)$ as candidates for extrema.

Segment QR: $y = 75, 0 \le x \le 100$:
Substitute $y = 75$ into the function f to obtain

$$f(x, 75) = 10x + 20(75) - 0.5x(75) = -27.5x + 1{,}500, \; 0 \le x \le 100.$$

Since the derivative of this function is $-27.5 \ne 0$, we find again that there are no critical points; only the endpoints $x = 0$ and 100. Since here $y = 75$, these give us $Q(0, 75)$ again and one new point $R(100, 75)$ as candidates for extrema.

Segment PR: $x = 100, 0 \le y \le 75$:
Substitute $x = 100$ into the function f to obtain

$$f(100, y) = 10(100) + 20y - 0.5(100)y = 1{,}000 - 30y, \; 0 \le y \le 75.$$

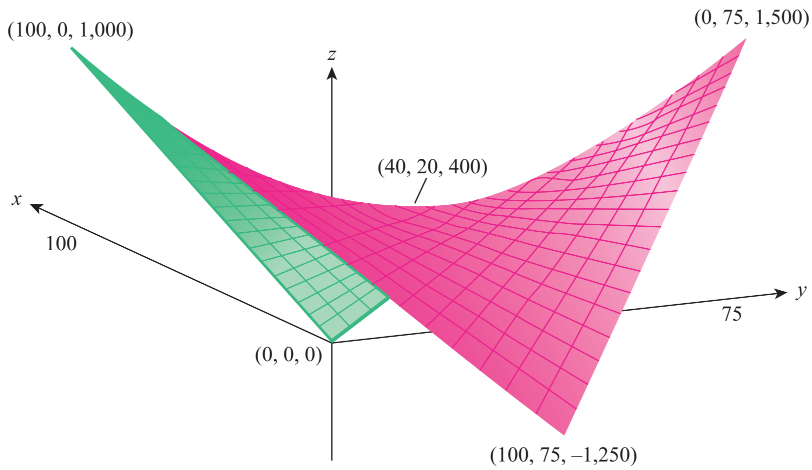
Again, there are no critical points, only the endpoints $y = 0$ and $y = 75$. These give us $P(100, 0)$ and $R(100, 75)$ again as candidates for extrema.

Our analysis has yielded the following five candidate points, shown here along with the values of f:

Point	Value of f	
(40, 20)	400	
(0, 0)	0	
(100, 0)	1,000	
(0, 75)	1,500	Absolute maximum
(100, 75)	−1,250	Absolute minimum

Since the absolute maximum *has* to be at one of these points, it must be at the point (0, 75), which yields a maximum monthly profit of \$1,500.

Before we go on... In Example 1 we did not classify the three points (40, 20), (0, 0), and (100, 0) as relative maxima, minima, or neither. While we could analyze the function further to obtain this information, it is easier to simply graph the function. Figure 4 shows the graph of $f(x, y) = 10x + 20y - 0.5xy$ as plotted on the Surface Grapher at the Website.



Technology format: `10*x+20*y-0.5*x*y`

Figure 4

From the figure, we see that f has a saddle point at the interior critical point (40, 20), a relative minimum at (0, 0), and a relative maximum at (100, 0).

We can restate the problem in Example 1 as a *constrained optimization problem* as follows:

Maximize $f = 10x + 20y - 0.5xy$	Objective function
subject to $0 \le x \le 100$	Inequality constraint
and $0 \le y \le 75$	Inequality constraint

If you have studied linear programming, this example should remind you of the problems you solved by that technique. However, the techniques of linear programming cannot, in general, be used to solve problems with nonlinear objective functions. ■

EXAMPLE 2 Absolute Extrema: Triangular Domain

Find the maximum and minimum value of $f(x, y) = 8 + xy - x - 2y$ on the triangular region R with vertices (0, 0), (2, 0), and (0, 4).

Solution The domain of f is the region R shown in Figure 5.

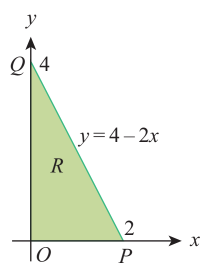


Figure 5

Step 1: Locate critical points in the interior of the domain.
We take the partial derivatives as usual.

$$f_x = y - 1 \qquad f_y = x - 2$$

Setting these partial derivatives equal to zero, we find that the only critical point is (2, 1). Since this lies outside the domain (the region R), we ignore it. Thus, there are no critical points in the interior of the domain of f.

Step 2: Locate extrema on the boundary of the domain.
The boundary of the domain consists of three line segments, *OP, OQ*, and *PQ*.

Segment OP: $y = 0$, $0 \le x \le 2$. The behavior of the function f along this segment is seen by substituting $y = 0$ into the expression for f:

$$f(x, 0) = 8 + x(0) - x - 2(0) = 8 - x$$

There are no critical points (the derivative of this function is never zero) and there are two endpoints, $x = 0$ and $x = 2$. Since $y = 0$, these endpoints give us the following two candidates for relative extrema: $O(0, 0)$ and $P(2, 0)$.

Segment OQ: $x = 0$, $0 \le y \le 4$. Along this segment we see

$$f(0, y) = 8 + (0)y - 0 - 2y = 8 - 2y.$$

Once again, there are no critical points, and only the endpoints $y = 0$ and $y = 4$. Since $x = 0$, this again gives us two candidates, $O(0, 0)$ and $Q(0, 4)$.

Segment PQ: This line segment has equation $y = 4 - 2x$ with $0 \le x \le 2$. Along this segment we see

$$\begin{aligned} f(x, 4 - 2x) &= 8 + x(4 - 2x) - x - 2(4 - 2x) \\ &= -2x^2 + 7x. \end{aligned}$$

Substitute $y = 4 - 2x$.

This function of x (whose graph is an upside-down parabola) has a maximum when its derivative, $-4x + 7$, is 0, which occurs when $x = 7/4$. When $x = 7/4$, $y = 4 - 2(7/4) = 1/2$. Thus, we have a critical point at $(7/4, 1/2) = (1.75, 0.5)$. The endpoints are $x = 0$ and $x = 2$, giving us $P(2, 0)$ and $Q(0, 4)$ once again.

If we compute the value of f at each candidate point, we obtain:

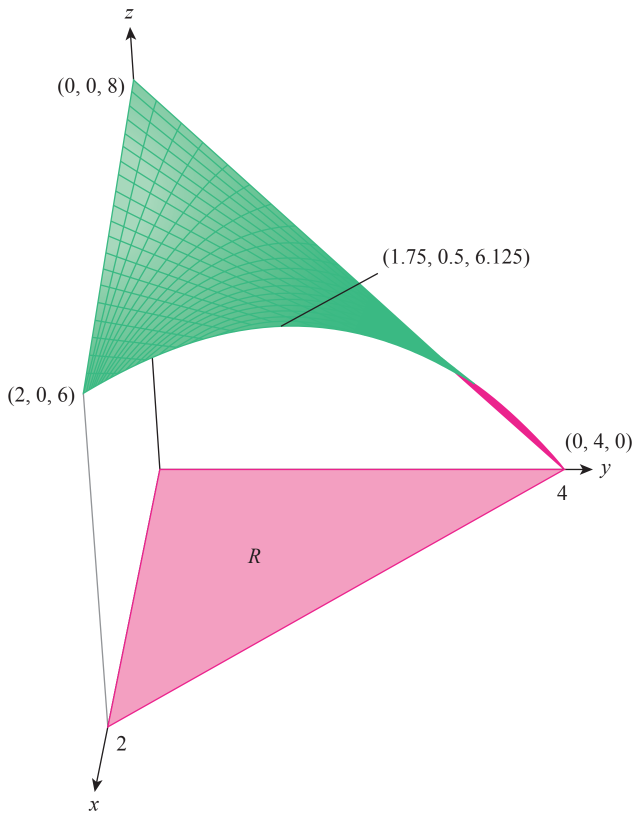
Point	Value of f	
(0, 0)	8	Absolute maximum
(2, 0)	6	
(0, 4)	0	Absolute minimum
(1.75, 0.5)	6.125	

We see that f has an absolute maximum of 8 at the point (0, 0) and an absolute minimum of 0 at the point (0, 4).

➡ **Before we go on...** Figure 6 shows the graph[1] of $f(x, y) = 8 + xy - x - 2y$ from Example 2, and tells us that f has a relative minimum at $P(2, 0)$. It also shows that f has neither a relative maximum nor a minimum at (1.75, 0.5)—it has a maximum at

[1] We sketched it on the Surface Grapher at the Website. To show the restriction to the triangular domain, we used the parametric surface feature with

```
x = u,  y = (4 − 2*u)*(1 − v),  z = 8 + u*(4 − 2*u)*(1 − v)
− u − 2*(4 − 2*u)*(1 − v) (0 ≤ u ≤ 2,  0 ≤ v ≤ 1)
```



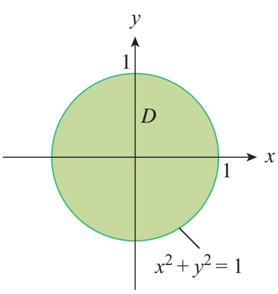
Graph of $f(x, y) = 8 + xy - x - 2y$
$0 \le x \le 2, 0 \le y \le 4 - 2x$

Figure 6

(1.75, 0.5) *along the edge PQ*, but the value $f(1.75, 0.75) = 6.125$ is smaller than the values of f at interior points immediately behind it. ■

EXAMPLE 3 Absolute Extrema: Circular Domain

Find the maximum and minimum value of $f(x, y) = x^2 + y^2 + y + 1$ subject to $x^2 + y^2 \le 1$.



$x^2 + y^2 \le 1$

Figure 7

Solution The domain D of the function f is the unit disc $\{(x, y) | x^2 + y^2 \le 1\}$ shown in Figure 7.

Step 1: Locate critical points in the interior of the domain.
We take the partial derivatives:

$$f_x = 2x \qquad\qquad f_y = 2y + 1$$

Setting these partial derivatives equal to zero, we find that the only critical point is $(0, -1/2)$, which is in the interior of D.

Step 2: Locate extrema on the boundary of the domain.
The boundary of D consists of all points (x, y) with $x^2 + y^2 = 1$. We can substitute this equation in the formula for f to obtain a function of a single variable:

$$f = (x^2 + y^2) + y + 1 = 2 + y$$

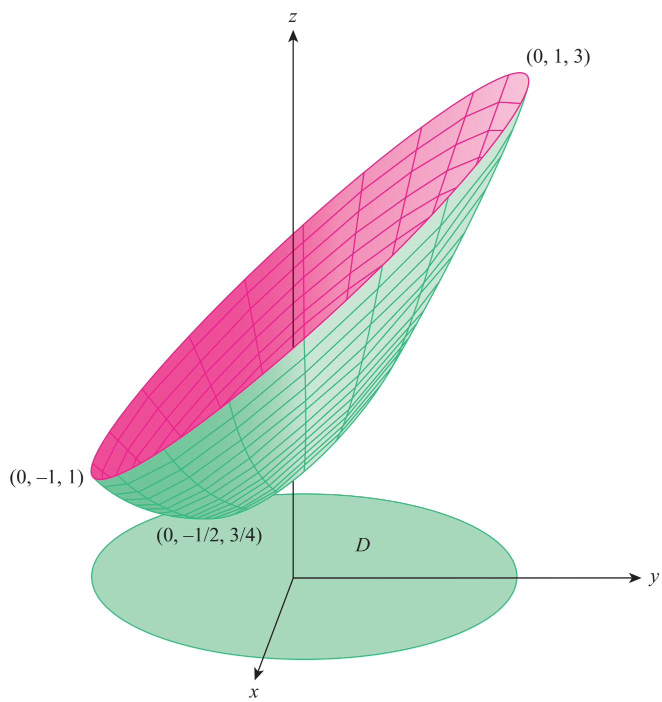
Notice that y cannot take an arbitrary value—since (x, y) is a point on the circle, we must have $-1 \le y \le 1$. This function of y has no critical points but does have two end points: 1 and -1. When $y = \pm 1$, the equation $x^2 + y^2 = 1$ tells us that $x = 0$. So, our candidate boundary points are $(0, -1)$ and $(0, 1)$.

The values of f at these candidate points are shown in the following table:

Point	Value of f	
$(0, -1/2)$	3/4	Absolute minimum
$(0, -1)$	1	
$(0, 1)$	3	Absolute maximum

Thus, f has an absolute maximum of 3 at $(0, 1)$ and an absolute minimum of 3/4 at $(0, -1/2)$.

Before we go on... Figure 8 shows the graph of $f(x, y) = x^2 + y^2 + y + 1$ from Example 3, and tells us that the point $(0, -1, 1)$ is not a relative extremum. (It is a



Graph of $f(x, y) = x^2 + y^2 + y + 1$
$x^2 + y^2 \le 1$

Figure 8

minimum along the boundary, but the value of *f* there is higher than its value at interior points immediately to its right.)

Notice also, that in locating extreme points on the boundary, we were actually solving the following constrained optimization problem:

Find the optimum values of $f(x, y) = x^2 + y^2 + y + 1$ subject to $x^2 + y^2 = 1$.

Here, the constraint takes the form of an equation rather than an inequality. Similarly, in the preceding examples, restricting to each of the various line segments amounted to solving an optimization problem with an equation constraint; for instance, in Example 2, on the segment *PQ,* we were finding the optimum values of $f(x, y) = 8 + xy - x - 2y$ subject to $y = 4 - 2x$. Solving optimization problems with equality constraints is discussed further in Section 15.4. ■

Some software packages, such as Excel, have built-in algorithms that seek absolute extrema with or without constraints. In the next example, we use the "Solver" add-in[2] in Excel to solve an optimization problem whose objective function has a more complicated domain.

EXAMPLE 4 **Solving an Optimization Problem Using Excel's Solver**

Use Excel's Solver to solve the following problem:

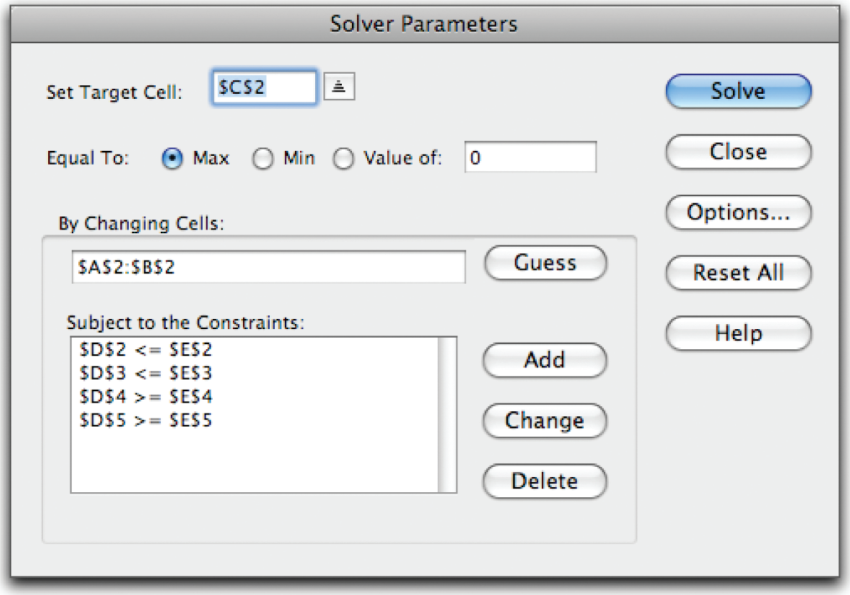
$$\begin{aligned} &\text{Maximize } P = 10x + 60y + 0.5xy && \text{Objective function} \\ &\text{subject to} \quad x + 5y \le 100 && \text{Constraint 1} \\ &\qquad 3x + 9y \le 270 && \text{Constraint 2} \\ &\qquad x \ge 0 \ \text{ and } \ y \ge 0 && \text{Constraints 3 and 4} \end{aligned}$$

Solution First, we set up the problem in spreadsheet form as follows:

	A	B	C	D	E	F
1	x	y	Objective	Constraints		
2	0	0	=10*A2+60*B2+0.5*A2*B2	=A2+5*B2	100	Constraint 1
3	Initial value of x	Initial value of y		=3*A2+9*B2	270	Constraint 2
4				=A2	0	Constraint 3
5				=B2	0	Constraint 4

A2 will be the cell that contains the value of x and B2 the cell that contains the value of y. Solver requires us to give them initial values, so we've set them both equal to 0; Solver will adjust them to find the optimal solution. Next, we select "Solver" in the "Tools" menu to bring up the Solver dialog box. Here is the dialog box with all the necessary fields completed to solve the problem:

[2] If "Solver" does not appear in the "Tools" menu, you should first install it using your Excel installation software. (Solver is one of the "Excel Add-Ins.")



Notes

- The Target Cell refers to the cell that contains the objective function.
- "Max" is selected because we are maximizing the objective function.
- "Changing Cells" are obtained by selecting the cells that contain the current values of x and y.
- Constraints are added one at a time by pressing the "Add" button and selecting the cells that contain the left- and right-hand sides of each inequality, as well as the type of inequality. (Equality constraints are also permitted.)

Once we've entered the parameters, we click on "Solve" and the (approximate) optimal solution appears in A2 and B2, with the maximum value of P appearing in cell C2. The optimal solution is therefore $x = 40$, $y = 12$, $P = 1,360$.

	A	B	C	D	E
1	x	y	Objective	Constraints	
2	40	12	1360	100	100
3	Final value of x	Final value of y		228	270
4				40	0
5				12	0

➡ Before we go on...

Q: *Can a software package such as Excel Solver be used interchangeably with the analytic method?*

A: Some optimization problems lead to equations that cannot be solved analytically, and so some form of numerical approach (such as that used in Solver) is essential in those cases. However, with all numerical approaches there is always a chance of running into one or more of these problems:

- The solution given will be a relative extremum rather than an absolute extremum.
- The solution is not exact.
- Roundoff errors lead to an incorrect solution or prevent finding a solution.
- Only one solution is given even if there is more than one absolute extremum.

So, use Solver with caution.

EXERCISES

▼ more advanced

T In all of the exercises for this section, a software package such as Excel Solver can be used as a check on your analytic work. (Bear in mind, however, the cautions at the end of Example 4.)

In Exercises 1–16, find the maximum and minimum values of the given function and the points at which they occur.

1. $f(x, y) = x^2 + y^2$; $0 \le x \le 2, 0 \le y \le 2$

2. $g(x, y) = \sqrt{x^2 + y^2}$; $1 \le x \le 2, 1 \le y \le 2$

3. $h(x, y) = (x - 1)^2 + y^2$; $x^2 + y^2 \le 4$

4. $k(x, y) = x^2 + (y - 1)^2$; $x^2 + y^2 \le 9$

5. $f(x, y) = e^{x^2+y^2}$; $4x^2 + y^2 \le 4$

6. $g(x, y) = e^{-(x^2+y^2)}$; $x^2 + 4y^2 \le 4$

7. $h(x, y) = e^{4x^2+y^2}$; $x^2 + y^2 \le 1$

8. $k(x, y) = e^{-(x^2+4y^2)}$; $x^2 + y^2 \le 4$

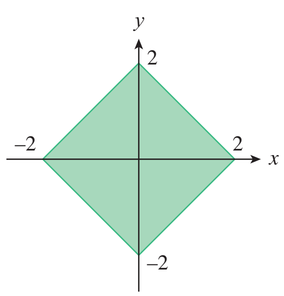
9. $f(x, y) = x + y + 1/(xy)$; $x \ge 1/2, y \ge 1/2, x + y \le 3$

10. $g(x, y) = x + y + 8/(xy)$; $x \ge 1, y \ge 1, x + y \le 6$

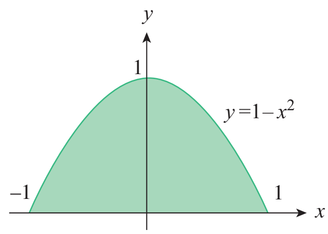
11. $h(x, y) = xy + 8/x + 8/y$; $x \ge 1, y \ge 1, xy \le 9$

12. $k(x, y) = xy + 1/x + 4/y$; $x \ge 1, y \ge 1, xy \le 10$

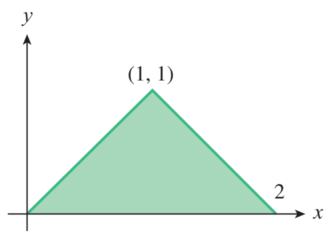
13. ▼ $f(x, y) = x^2 + 2x + y^2$; on the region in the figure



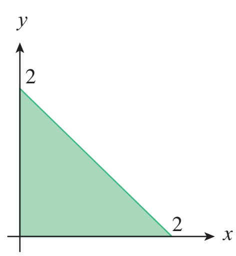
14. ▼ $g(x, y) = x^2 + y^2$; on the region in the figure



15. ▼ $h(x, y) = x^3 + y^3$; on the region in the figure



16. ▼ $k(x, y) = x^3 + 2y^3$; on the region in the figure



APPLICATIONS

17. ***Cost*** Your bicycle factory makes two models, five-speeds and ten-speeds. Each week, your total cost (in dollars) to make x five-speeds and y ten-speeds is

$$C(x, y) = 10{,}000 + 50x + 70y - 0.5xy$$

You want to make between 100 and 150 five-speeds and between 80 and 120 ten-speeds. What combination will cost you the least? What combination will cost you the most?

18. ***Cost*** Your bicycle factory makes two models, five-speeds and ten-speeds. Each week, your total cost (in dollars) to make x five-speeds and y ten-speeds is

$$C(x, y) = 10{,}000 + 50x + 70y - 0.46xy$$

You want to make between 100 and 150 five-speeds, and between 80 and 120 ten-speeds. What combination will cost you the least? What combination will cost you the most?

19. ▼ ***Profit*** Your software company sells two operating systems, Walls and Doors. Your profit (in dollars) from selling x copies of Walls and y copies of Doors is given by

$$P(x, y) = 20x + 40y - 0.1(x^2 + y^2).$$

If you can sell a maximum of 200 copies of the two operating systems together, what combination will bring you the greatest profit?

20. ▼ ***Profit*** Your software company sells two operating systems, Walls and Doors. Your profit (in dollars) from selling x copies of Walls and y copies of Doors is given by

$$P(x, y) = 20x + 40y - 0.1(x^2 + y^2).$$

If you can sell a maximum of 400 copies of the two operating systems together, what combination will bring you the largest profit?

21. ***Temperature*** The temperature at the point (x, y) on the square with vertices (0, 0), (0, 1), (1, 0), and (1, 1) is given by $T(x, y) = x^2 + 2y^2$. Find the hottest and coldest points on the square.

22. ***Temperature*** The temperature at the point (x, y) on the square with vertices (0, 0), (0, 1), (1, 0), and (1, 1) is given by $T(x, y) = x^2 + 2y^2 - x$. Find the hottest and coldest points on the square.

23. ▼ ***Temperature*** The temperature at the point (x, y) on the disc $\{(x, y) | x^2 + y^2 \leq 1\}$ is given by $T(x, y) = x^2 + 2y^2 - x$. Find the hottest and coldest points on the disc.

24. ▼ ***Temperature*** The temperature at the point (x, y) on the disc $\{(x, y) | x^2 + y^2 \leq 1\}$ is given by $T(x, y) = 2x^2 + y^2$. Find the hottest and coldest points on the disc.

Extreme Values: Answers to Odd-Numbered Exercises

1. Maximum value of 8 at (2, 2), minimum value of 0 at (0, 0)

3. Maximum value of 9 at (−2, 0), minimum value of 0 at (1, 0)

5. Maximum value of e^4 at (0, ±2), minimum value of 1 at (0, 0)

7. Maximum value of e^4 at (±1, 0), minimum value of 1 at (0, 0)

9. Maximum value of 5 at (1/2, 1/2), minimum value of 3 at (1, 1)

11. Maximum value of 161/9 at (1, 9) and (9, 1), minimum value of 12 at (2, 2)

13. Maximum value of 8 at (2, 0), minimum value of −1 at (−1, 0)

15. Maximum value of 8 at (2, 0), minimum value of 0 at (0, 0)

17. For minimum cost of $16,600, make 100 5-speeds and 80 10-speeds. For maximum cost of $17,400, make 100 5-speeds and 120 10-speeds.

19. For a maximum profit of $4,500, sell 50 copies of Walls and 150 copies of Doors.

21. Hottest point: (1, 1), coldest point: (0, 0)

23. Hottest points: $(-1/2, \pm\sqrt{3}/2)$, coldest point: (1/2, 0)

EXTREME VALUES: ANSWERS TO ODD-NUMBERED EXERCISES