SOLUTION TO PROBLEM 46 OF CHAPTER 7

Let $X_k = 1$ or 0 according as the kth bolt is defective or not. The random variables X_1, X_2, ... are independent and identically distributed Bernoulli random variable with parameter $p = 0.05$ (i.e., binomial with parameters $n = 1$ and $p = 0.05$). So their common mean and variance are $\mu = 0.05$ and $\sigma^2 = 0.05 \cdot 0.95 = 0.0475$.

Let $S_n = X_1 + X_2 + \cdots + X_n$ and let X denote the total number of defective bolts in a shipment of 10,000. Then $X = S_{10000}$. We want to find the smallest integer a so that $P(X > a) \leq 0.01$ or, equivalently, $P(X \leq a) \geq 0.99$. Using the normal approximation with continuity correction [(17) on page 189], we get that

$$P(X \leq a) = P(S_{10000} \leq a) \approx \Phi\left(\frac{a + \frac{1}{2} - 10000 \cdot 0.05}{\sqrt{0.0475}\sqrt{10000}}\right) = \Phi\left(\frac{a - 499.5}{\sqrt{475}}\right)$$

Using Table I on page 253, we find that $\Phi(2.33) = 0.99$ (approximately). Therefore, we want $(a - 499.5)/\sqrt{475} \geq 2.33$. The smallest integer satisfying this inequality is $a = 551$. This completes the solution. Compare the result given here to the one obtained using Chebychev's inequality in Exercise 27 of Chapter 4.

The Exact Solution

The total number, X, of defective bolts in a shipment of 10,000 has exactly the binomial distribution with parameters $n = 10000$ and $p = 0.05$. Therefore, the exact solution to the problem is the smallest integer a such that

$$\sum_{x=0}^{a} \binom{10000}{x} (0.05)^x (0.95)^{10000-x} \geq 0.99.$$

Below is a BASIC-type program for obtaining a. It is based on the fact that the discrete density function, f, of a binomial distribution with parameters n and p satisfies the equation

$$f(x) = \left(\frac{p}{1-p}\right)\left(\frac{n-x+1}{x}\right) f(x-1),$$

for $x = 1, 2, \ldots, n$. Below is the program. Note that you will need a good calculating device to run this program properly. We ran it and got `A = 551`, the same result as given by the normal approximation.

```
 10 LET P=0.05
 20 LET N=10000
 30 LET Q=P/(1-P)
 40 LET F=(1-P)^N
 50 LET T=F
 60 FOR X=1 TO N
 70 LET F=Q*(N-X+1)/X*F
 80 LET T=T+F
 90 IF T>=0.99 THEN GOTO 110
100 NEXT X
110 PRINT "A = ";X
120 END
```