From Polya's "How to Solve It."

A rate problem. *Water is flowing into a conical vessel at the rate r. The vessel has the shape of a right circular cone, with horizontal base, the vertex pointing downwards; the radius of the base is a, the altitude of the cone b. Find the rate at which the surface is rising when the depth of the water is y. Finally, obtain the numerical value of the unknown supposing that* $a = 4$ *ft.,* $b = 3$ *ft.,* $r = 2$ *cu. ft. per minute, and* $y = 1$ *ft.*

The students are supposed to know the simplest rules of differentiation and the notion of "rate of change."

"What are the data?"

"The radius of the base of the cone $a = 4$ ft., the altitude of the cone $b = 3$ ft., the rate at which the water is flowing into the vessel $r = 2$ cu. ft. per minute, and the depth of the water at a certain moment, $y = 1$ ft."

"Correct. The statement of the problem seems to suggest that you should disregard, provisionally, the numerical values, work with the letters, express the unknown in terms of a, b, r, y and only finally, after having obtained the expression of the unknown in letters, substitute the numerical values. I would follow this suggestion. Now, *what is the unknown?"*

"The rate at which the surface is rising when the depth of the water is y."

"What is that? Could you say it in other terms?"

"The rate at which the depth of the water is increasing."

"What is that? *Could you restate it still differently?"*

"The rate of change of the depth of the water."

"That is right, the rate of change of y. But what is the rate of change? *Go back to the definition."*

"The derivative is the rate of change of a function."

"Correct. Now, is y a function? As we said before, we disregard the numerical value of y. Can you imagine that y changes?"

"Yes, y, the depth of the water, increases as the time goes by."

"Thus, y is a function of what?"

"Of the time t."

"Good. *Introduce suitable notation.* How would you write the 'rate of change of y' in mathematical symbols?"

"$\frac{dy}{dt}$"

"Good. Thus, this is your unknown. You have to express it in terms of a, b, r, y. By the way, one of these data is a 'rate.' Which one?"

"r is the rate at which water is flowing into the vessel."

"What is that? Could you say it in other terms?"

"r is the rate of change of the volume of the water in the vessel."

"What is that? *Could you restate it still differently?* How would you write it in *suitable notation?"*

"$r = \frac{dV}{dt}$."

"What is V?"

"The volume of the water in the vessel at the time t."

"Good, Thus, you have to express $\frac{dy}{dt}$ in terms of a, b, $\frac{dV}{dt}$, y. How will you do it?"

...

"If you cannot solve the proposed problem try to solve first some related problem. If you do not see yet the connection between $\frac{dy}{dt}$ and the data, try to bring in some simpler connection that could serve as a stepping stone.

...

"Do you not see that there are other connections? For instance, are y and V independent of each other?"

"No. When y increases, V must increase too."

"Thus, there is a connection. What is the connection?"

"Well, V is the volume of a cone of which the altitude is y. But I do not know yet the radius of the base."

"You may consider it, nevertheless. Call it something, say x."

"$V = \frac{\pi x^2 y}{3}$."

"Correct. Now what about x. Is it independent of y?"

"No. When the depth of the water, y, increases the radius of the free surface, x, increases too."

"Thus, there is a connection. What is the connection?"

"Of course, similar triangles.

$$x : y = a : b."$$

"One more connection, you see. I would not miss profiting from it. Do not forget, you wished to know the connection betwen V and y."

"I have

$$x = \frac{ay}{b}$$
$$V = \frac{\pi a^2 y^3}{3^{b^2}}."$$

"Very good. This looks like a stepping stone, does it not? But you should not forget your goal. *What is the unknown?"*

"Well, $\frac{dy}{dt}$."

"You have to find a connection betwen $\frac{dy}{dt}$, $\frac{dV}{dt}$, and other quantities. And here you have one between y, V, and other quantities. What to do?"

"Differentiate! Of course!

$$\frac{dV}{dt} = \frac{\pi a^2 y^2}{b^2} \frac{dy}{dt}.$$

Here it is."

"Fine! And what about the numerical values?"

"If $a = 4$, $b = 3$, $\frac{dV}{dt} = r = 2$, $y = 1$, then

$$2 = \frac{\pi \times 16 \times 1}{9} \frac{dy}{dt}."$$