GRAPHING ACCURATELY

We show here all the steps you should follow to graph a function by working out in detail an example for the function

$$y = f(x) = \frac{-x^2}{(2-x)^2}.$$

1. Find the domain of the function. We need to omit all values of x which result in division by zero, or roots of even index of negative numbers, or logarithms of negative numbers or zero.

In our example, we need to make the denominator different from zero: $(2 \Leftrightarrow x)^2 \neq 0 \Leftrightarrow 2 \Leftrightarrow x \neq 0 \Leftrightarrow x \neq 2$. In interval notation, the domain is $(\Leftrightarrow\infty, 2) \cup (2, \infty)$.

2. Find the intercepts of the graph, but setting one variable equal to zero and solving for the other one. For $x = 0$ we have $y = f(0) = \frac{-0^2}{(2-0)^2} = 0$, giving the point (0,0). Similarly, for $y = 0$, we have $y = \frac{-x^2}{(2-x)^2} = 0$ which implies $\Leftrightarrow x^2 = 0$ and thus $x = 0$, giving once more the point (0,0).

3. Find the intervals of positivity and negativity (if possible). In this case we have $f(x) = \Leftrightarrow\frac{x^2}{(2-x)^2} \leq 0$ whenever the expression is defined (since squares are never negative). This means that the graph of the function lies entirely below the x-axis or on it.

4. Find the vertical asymptotes (if any). These are the lines $x = c$ where the numerator does not vanish for $x = c$ but the denominator does. In this case we have the denominator becoming zero for $x = 2$ only, and the numerator does not vanish for $x = 2$. Thus, $x = 2$ is a vertical asymptote.

5. (Use of the derivative) Find intervals where the function increases, where it decreases, and local extrema. We compute the first derivative

$$f'(x) = \frac{-2x(2-x)^2 - (-x^2)2(2-x)(-1)}{(2-x)^4} = -4x(2-x)^{-3}.$$

We now see that $f'(x) > 0$ for $x < 0$ and for $x > 2$; $f'(x) < 0$ for $0 < x < 2$. Thus the function is increasing in the intervals (-∞,0) and (2,∞), and decreasing on the interval (0,2). At $x = 0$, $y = f(0) = 0$, (that is, at the point (0,0)) there is a local maximum, since the function changes from increasing to decreasing. BEWARE: At $x = 2$ there is no local extremum since the function is not even defined there.

6. (Use of the second derivative) Find intervals where the graph is concave up or down, and inflection points. We compute the second derivative

$$f''(x) = (-4)(2-x)^{-3} + (-4x)(-3)(2-x)^{-4}(-1) = -8(x+1)(2-x)^{-4}.$$

We now see that $f''(x) > 0$ for $x < \Leftrightarrow 1$; $f''(x) < 0$ for $\Leftrightarrow 1 < x$. Thus the graph of the function is concave upward in the intervals (-∞,-1) and concave downward on the intervals (-1,2) and (2,∞). At $x = \Leftrightarrow 1$, $y = f(\Leftrightarrow 1) = \Leftrightarrow\frac{1}{9}$, (that is, at the point $(\Leftrightarrow 1, \frac{1}{9})$) there is an inflection point, since the graph changes from concave upward to concave downward.

7. Find the limits at infinity. We compute in this case $\lim_{x\to\pm\infty} \frac{-x^2}{(2-x)^2} = \lim_{x\to\pm\infty} \frac{-1}{(\frac{2}{x}-1)^2} = \Leftrightarrow 1$. Since the limits at infinity are finite, there are horizontal asymptotes where y equals these limits. In this case, we have a horizontal asymptote at the line $y = \Leftrightarrow 1$.

8. Find the left and right limits as x approaches each number where there is a "break" in the domain of f. In this example, the only break in the domain of f occurs at $x = 2$. So, we must compute the limits

$$L = \lim_{x\to 2^-} f(x) \text{"="} \frac{-(-2^-)^2}{(2-2^-)^2} \text{"="} \frac{-4}{(0^+)^2} \text{"="} -\frac{4}{0^+} = \infty, \quad \text{and} \quad R = \lim_{x\to 2^+} f(x) \text{"="} \frac{-(-2^+)^2}{(2-2^+)^2} \text{"="} \frac{-4}{(0^-)^2} \text{"="} -\frac{4}{0^+} = \infty.$$

If one or both of these limits had been finite, we would draw **hollow** *points at* (2,L) *and/or at* (2,R), *from which discontinuous branches of the graph would originate.*

9. Plot all this information. Any points you found as intercepts (item 2), local extrema (item 5), inflection points (item 6), and points of discontinuity (item 8) should be plotted exactly at this point. Then draw dotted lines where the vertical and horizontal asymptotes are (items 4 and 7, if any). Finally join smoothly all those points respecting intervals of positivity and negativity (item 3), intervals of increase and decrease (item 5), intervals of concavity upward and downward (item 6), and limits at points of discontinuity and at infinity (items 7 and 8).