The Fundamental Theorem of Calculus (FTC)

Let f be a continuous function on an interval I and let c be a fixed number in I. **For any** $x \in I$ **define**

$$y = F(x) = \int_c^x f(t)\,dt.$$

Then, $F'(x) = \dfrac{dy}{dx} = f(x)$ **for all** $x \in I$**.**

That is, F is an antiderivative of f on I.

Remark: If $f(t) \geq 0$ for t in I, then an antiderivative of f is the *area function* that measures the area of the region bounded by the graph of f, the horizontal axis, and the vertical lines $t = c$ and $t = x$.

Example: Let $g(x) = \int_{-\pi}^{x} \cos(t^2)\, dt$. Find $g'(x)$.

Solution: *Before we can use the FTC*, we note that the lower limit of integration is constant, $c = -\pi$, and the upper limit is the variable with respect to which we want to differentiate, x, as required by the FTC. The integrand is $f(t) = \cos(t^2)$. Then, the FTC says $g'(x) = f(x) = \cos(x^2)$.

Example: Let $\phi(x) = \int_x^{7e} \ln(\tan(2t))\, dt$. Find $\phi'(x)$.

Solution: Since the *upper* limit of integration is constant and the *lower* limit is the variable with respect to which we want to differentiate, we rewrite the given function as

$$\phi(x) = -\int_{7e}^{x} \ln(\tan(2t))\, dt = \int_{7e}^{x} -\ln(\tan(2t))\, dt$$

$$= \int_{7e}^{x} \ln\left(\frac{1}{\tan(2t)}\right) dt = \int_{7e}^{x} \ln(\cot(2t))\, dt$$

Then, the FTC says $\phi'(x) = \ln(\cot(2x))$.

Example: Let $y = \int_2^{x^2} e^t \, dt$. Find $\dfrac{dy}{dx}$.

Solution: *Before we can use the FTC*, we note that the lower limit of integration is constant, $c = 2$, but the upper limit is *not* the variable with respect to which we want to differentiate, x, as required by the FTC. So, we introduce another variable, $u = x^2$, and write $y = \int_2^u e^t \, dt$. Then, according to the chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$. Now the FTC says $\dfrac{dy}{du} = e^u$. Then, $\dfrac{dy}{dx} = (e^u)(2x) = 2xe^{x^2}$.

Example: Let $y = \displaystyle\int_{g(x)}^{h(x)} f(t)\,dt$. Find $\dfrac{dy}{dx}$.

Solution: *Before we can use the FTC*, we must have integrals with the lower limit of integration constant and the upper limit equal to the variable with respect to which we want to differentiate. So, we introduce two other variables, $u = g(x)$ and $v = h(x)$, and write $y = \displaystyle\int_u^v f(t)\,dt = \int_u^0 f(t)\,dt + \int_0^v f(t)\,dt = -\int_0^u f(t)\,dt + \int_0^v f(t)\,dt$. Then, using the chain rule and the FTC,

$$\frac{dy}{dx} = \frac{d}{dx}\left(-\int_0^u f(t)\,dt\right) + \frac{d}{dx}\left(\int_0^v f(t)\,dt\right)$$

$$= -\frac{du}{dx}\,\frac{d}{du}\left(\int_0^u f(t)\,dt\right) + \frac{dv}{dx}\,\frac{d}{dv}\left(\int_0^v f(t)\,dt\right)$$

$$= -g'(x)\,f\left(g(x)\right) + h'(x)\,f\left(h(x)\right)$$

Example: Let $y = \displaystyle\int_{e^x}^{-x^2} \sin t \, dt$. Find $\dfrac{dy}{dx}$.

Solution: *Before we can use the FTC*, we rewrite y as in the previous example:

$$y = -\int_0^{e^x} \sin t \, dt + \int_0^{-x^2} \sin t \, dt$$

Here $f(t) = \sin t$, $g(x) = e^x$, and $h(x) = -x^2$. Then,

$$\frac{dy}{dx} = -g'(x)\, f\left(g(x)\right) + h'(x)\, f\left(h(x)\right)$$

$$= -e^x \sin(e^x) - 2x \sin(-x^2)$$

$$= 2x \sin(x^2) - e^x \sin(e^x).$$